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MODERN PLANE GEOMETRY

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THE PROOFS OF THE THEOREMS IN THE SYLLABUS OF MODERN PLANE GEOMETRY ISSUED BY THE ASSOCIATION FOR THE IMPROVEMENT OF GEOMETRICAL TEACHING

WITH THE SANCTION OF THE COUNCIL OF THE A.I.G.T.

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PREFACE

THE following Treatise is intended to be an introduction to the subject of Modern Plane Geometry and to the more advanced books of Cremona and others. It has a twofold object: to serve, in the first place, as a sequel to Euclid, or rather to the Treatise on Elementary Plane Geometry issued by the A.I.G.T.; and secondly, as a systematic means of procedure from Euclidean Geometry to the higher descriptive Geometry of Conics and of imaginary points. The first seven chapters, in the main, are intended to carry out the first object, and treat of the Geometry of the Triangle, Quadrangle and Circle, Harmonic Ratio (Chap. III.), and also (Chap. VII.) of Geometrical Maxima and Minima. In Chap. II. the triangle is very fully treated, and it is hoped that a sufficient introduction is given to the more recent researches on special points and circles associated with a triangle. Conic sections other than the circle are not introduced till Chap. VIII., Theorem 7, after which a knowledge of elementary Geometrical Conics is assumed.

The theory of Projections is such a necessary part

of Descriptive Plane Geometry, and at the same time has so little to do with Solid Geometry, that no apology is needed on behalf of the A.I.G.T. for introducing the subject in the Syllabus of Modern *Plane* Geometry, nor on behalf of the Authors for retaining the title of the Syllabus.

There has been practically no departure from the Syllabus of the A.I.G.T., the whole of which is here reproduced; the numbering of the Theorems has been retained, although a few important additions have been made; and collections of examples and problems have been appended to each chapter.

The Authors desire to express their thanks to the Council of the A.I.G.T. for permission to make the Syllabus the basis of a text-book, and also to the members of the Committee on Modern Plane Geometry, who drew up the Syllabus, and to whom any merit this book may possess is due. Their thanks are also tendered to Dr. Taylor, Master of St. John's College, Cambridge, for permission to use his proof of Theorem 3, Chap. IV., and also for much intangible though invaluable aid contained in his standard work on the *Ancient and Modern Geometry of Conics*. They are indebted to friends —in particular, to Mr. T. P. Kent, B.A., former scholar of Christ Church—for looking through the proof-sheets as the work has passed through the press, and they trust it may be free from many misprints.

G. Richardson.
A. S. Ramsey.

Dec. 1893.

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MODERN PLANE GEOMETRY

CHAPTER I

INTRODUCTION

GEOMETRY may be divided into two kinds, metrical and descriptive. The former treats of the lengths of lines and the magnitudes of angles and areas; whilst the latter has to do with the positions, directions, and intersections of lines, the *loci* of points, and the nature and character of curves. The proposition—"the sum of the squares on the sides containing the right angle of a right-angled triangle is equal to the square on the side subtending the right angle"—is an instance of a metrical theorem; "the line which joins the centres of two circles which touch each other passes through the point of contact"—is an instance of a descriptive theorem.

The geometry of Euclid and the Ancients is almost entirely metrical; and one of the noticeable features of modern geometry as contrasted with ancient geometry is that on the whole it is descriptive rather than metrical.

There are, however, two great principles which under-

lie modern plane geometry, and distinguish it from ancient geometry or the Euclidian methods. These are the *principle of continuity* and the *principle of duality*. By the *principle of continuity* is meant that a theorem should be of universal truth and application, and should be involved by no exceptions. If altered circumstances seemingly make the theorem untrue, the circumstances must give way and not the theorem—that is, suitable conventions must be introduced in interpreting the circumstances, so that the enunciation of the theorem is not invalidated. The conventions which are thus necessarily introduced for the establishment of this principle are the ideas involved in the algebraic symbols + and − as affecting the direction in which lines are measured, and the position of areas, and the transcendental notions of imaginary points and lines. It is important that the student should observe how the principle is gradually developed as he proceeds with the subject, and he will find it a useful exercise to see how it applies in theorems which apparently do not come within its scope. The *principle of duality* will be explained later (see Chap. IV.) It will be sufficient to say of it here that to every descriptive theorem there exists another theorem (its dual), so related to it that to every point in the former there corresponds a straight line in the latter, and to every straight line in the former a point in the latter.

PROPERTIES OF A STRAIGHT LINE

(1) In Elementary Geometry a straight line of definite

length, or a SEGMENT, is considered with reference to absolute length only, and the question of whether it is to be added to or subtracted from another line is a matter of explicit statement.

In Modern Plane Geometry account is taken of the direction of measurement along a straight line, and this is expressed with the help of the signs + and − .

The fundamental rules with regard to these signs are that

[i] if A and B be any two points on a straight line,

$$AB = -BA, \quad BA = -AB;$$

an immediate consequence of which is that

$$-(-AB) = AB;$$

[ii] if A, B, P be any three points on a straight line,

$$AP + PB = AB.$$

(2) Two fixed points (A and B) being taken on a straight line, the position of any other point P (on the same straight line) is given when the ratio of the segments AP : PB is given.

If AP : PB is positive, AB is internally divided in P.

If AP : PB is negative, AB is externally divided in P.

In either case P is nearer to A or to B, according as AP : PB numerically < or > unity.

If $\frac{AP}{PB} = +1$, then P bisects AB.

If $\frac{AP}{PB} = -1$, then P is at an infinite distance.

The point at an infinite distance beyond A is identically the same point as that at an infinite distance beyond B; for

when P is at A, AP : PB is zero;

when P is at the middle point of AB then AP : PB is $+1$,

as P approaches B the ratio continually increases in value,

becoming equal in the limit to $+\infty$;

as P passes through B the sign of the ratio changes and the ratio gradually decreases numerically from $-\infty$ (in the limit) to -1, which is the value it assumes when P is at an infinite distance beyond B.

Up to this point the value of the ratio has undergone a continuous change, and the motion of P has also been continuous. As P moves from the point at an infinite distance beyond A to A, the ratio gradually decreases numerically in value from -1 till in the limit it assumes the value zero. The changes in the value of the ratio have been continuous throughout. This, then, points to the conclusion that the motion of P has also been continuous; that is, that the point at an infinite distance beyond B may be considered as identically the same as that at an infinite distance beyond A, and that the straight line may be considered as a continuous locus.

NOTE.—When a straight line AB is mentioned, a confusion is likely to arise in the mind of the student as to whether the *finite* straight line AB is meant, or

the continuous *locus* of which $\mathring{A}B$ is a portion. It may be safely inferred that the latter is always meant except where the length AB is expressed or implied.

(3) Ratios may be compounded to any extent in the way of either multiplication or division. It is usual to write

$$\frac{a}{p}\cdot\frac{b}{q}\cdot\frac{c}{r}\cdot\frac{d}{s}\cdot\frac{e}{t}=1, \text{ as } abcde=pqrst;$$

$$\frac{AB}{AC}+\frac{AB}{AD}=2, \text{ as } \frac{1}{AC}+\frac{1}{AD}=\frac{2}{AB},$$

and so forth ; but it must be distinctly understood that the second equations are simply convenient modes of representing the first, and are not to be taken as implying that there is any geometrical meaning assignable to the quantities *abcde*, $\frac{2}{AB}$, etc., taken separately.

(4) The following are illustrations of relations between the segments of a straight line :—

[i] If A, B, C be three points in a straight line,

$$AB + BC + CA = 0.$$

This is obvious.

[ii] If P be any other point in the straight line,

$$PA \,.\, BC + PB \,.\, CA + PC \,.\, AB = 0$$

A B C P

For $PA \,.\, BC + PB \,.\, CA + PC \,.\, AB =$

$$\begin{aligned} &- AP(AC - AB) - (AB - AP)AC + (AC - AP)AB \\ &= - AP \,.\, AC + AP \,.\, AB - AB \,.\, AC + AP \,.\, AC \\ &\quad + AC \,.\, AB - AP \,.\, AB = 0. \end{aligned}$$

[iii] $PA^2 . BC + PB^2 . CA + PC^2 . AB = -BC . CA . AB.$

The left-hand member

$= AP^2(AC - AB) - (AB - AP)^2 AC + (AC - AP)^2 AB$

$= AP^2 . AC - AP^2 . AB - AB^2 . AC + 2AP . AB . AC$

$\quad - AP^2 . AC + AC^2 . AB - 2AP . AC . AB + AP^2 . AB$

$= AB . AC . (AC - AB) = AB . AC . BC$

$= -BC . CA . AB.$

[iv] If P bisects BC, $AP = \frac{1}{2}(AB + AC)$.

For $BP = -CP;$

$\therefore AP - AB = -(AP - AC),$

or $2AP = AB + AC.$

[v] If A, B, C, D be four points in a straight line, and P bisect AB and Q bisect CD,

$$PQ = \tfrac{1}{2}(AC + BD) = \tfrac{1}{2}(AD + BC).$$

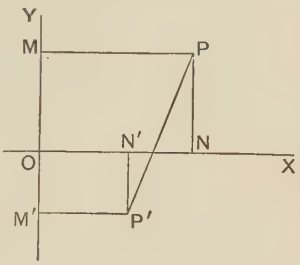
For $2PQ = 2PB + 2BC + 2CQ$

$= AB + BC + BC + CD$

$= AC + BD,$ or $AD + BC.$

(5) It will have been observed that in the figure for the previous theorems we have considered the segments of the line when measured from the left to the right to be positive. This, although usual, is quite arbitrary, the essential point being that whatever direction be chosen as the positive direction of measurement, then the *opposite* direction must be negative. The term *sense* is used to imply that this consideration is adopted; thus AB and BA denote the same segment taken in *opposite senses*. The consideration of *sense* has only been

applied to segments of the same straight line, but it may be extended to segments of parallel straight lines, thus: if from two points P, P′ perpendiculars be drawn to two straight lines OX, OY at right angles to each other, then PN, P′N′ have opposite senses, whilst PM, P′M′ have the same sense; and it follows from this



that $PN - P'N' = MM'$,

just as $PM - P'M' = NN'$.

(6) DEFS.—The segment of the straight line intercepted between two points is called the JOIN of the points.

The foot of the perpendicular from any point on a straight line is called the PROJECTION of the point on the line.

The PROJECTION OF A SEGMENT OF A STRAIGHT LINE on a given line is the join of the projections of the extremities of the segment. Thus the projections of PP′ (see last fig.) on OX and OY are NN′ and MM′ respectively.

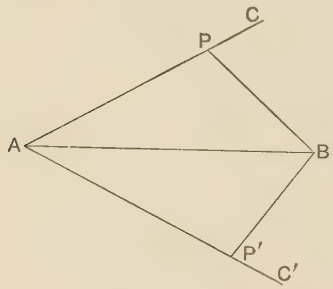
It is easily seen that the sum of the projections of the sides, taken in order, of any rectilinear figure on any straight line is zero.

(7) The convention of sign may be extended to angles; thus if we suppose that the angle BAC is generated by a straight line starting from AB and

turning about A, then the angles BAC, BAC′ have opposite senses. Also BAC and CAB have opposite senses, so that

$$\text{BAC} + \text{CAB} = 0.$$

As the idea of direction is not necessarily involved in the generation of plane figures or areas, the convention of signs cannot always be applied to areas, but whenever direction is involved in the description of an area, then the convention of signs does apply. Thus if two triangles ABP, ABP′ or other figures be described on *opposite* sides of the same line AB, then the figures ABP, ABP′ have opposite senses. ABP and BAP may also be regarded as the same area taken in opposite senses, so that ABP + BAP = 0.



CENTRE OF MEAN POSITION

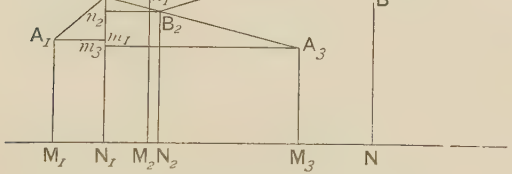
DEF.—If there be any number of points, say n, distributed in any manner in a plane, and a point be so determined that n times its distance from any straight line in the plane is always equal to the sum of the distances of the given points from the same straight line, that point is called the CENTRE OF MEAN POSITION or CENTROID of the system.

To determine the centre of mean position in any particular case we may proceed as follows :—

(1) Take any two intersecting straight lines in the plane, and draw lines parallel to them, each distant $\frac{1}{n}$th of the sum of the distances of the n points of the system from the corresponding line, and where these lines meet will be the centre of mean position.

(2) Or, if A_1, A_2, A_3, $\cdots A_n$ be the points, join A_1A_2 and bisect it at B_1 ; join B_1A_3 and divide it at B_2, so that $2B_1B_2 = B_2A_3$; join B_2A_4 and divide it at B_3, so that $3B_2B_3 = B_3A_4$; and so on till all the n points have been connected, and the last point of division B will be the centre of mean position. That the same point is determined by this method as by the first may be shown thus :—

Draw perpendiculars A_1M_1, A_2M_2, etc., B_1N_1, B_2N_2, etc., to any line in the plane ; also draw A_1m_1, B_1n_1, B_2n_2,



A_3m_3, etc., parallel to this line as in the figure. Then from equal triangles

$$A_2n_1 = B_1m_1 ;$$

$$\therefore\ 2B_1N_1 = A_1M_1 + A_2M_2\,;$$

also by parallels $B_1n_2 : n_2m_3 = B_1B_2 : B_2A_3\,;$

$$\therefore\ 2B_1n_2 = n_2m_3,\ \therefore\ 3B_2N_2 = 3n_2N_1 + 2B_1n_2 - n_2m_3$$
$$= 2B_1N_1 + A_3M_3$$
$$= A_1M_1 + A_2M_2 + A_3M_3,$$

and so on. In this manner we get

$$n\,.\,BN = A_1M_1 + A_2M_2 + A_3M_3 + \cdots + A_nM_n.$$

Moreover, it is evident from this result that it is immaterial in what order the given points are joined, as the same point B will always be finally attained.

The following results are evident from the definition :—

(1) If A, B, C, D, etc., be n collinear points, O their centre of mean position, and P any other point in the same straight line,

$$AP + BP + CP + DP + \text{etc.} = n\,.\,OP,$$

and
$$AO + BO + CO + DO + \text{etc.} = 0.$$

(2) The same results are true if the system of points are not collinear, and A, B, C, D, etc., be their projections on any straight line, O the projection of their centre of mean position, and P any point on the same straight line.

Examples on Chapter I

1. Prove that Theorem (4) [iii.] is also true when the point P is outside the line ABC.

2. If A, B, C, D be four collinear points and P any other point, prove that

$$PA^2 . BC . CD . DB - PB^2 . CD . DA . AC + PC^2 . DA . AB . BD - PD^2 . AB . BC . CA = 0.$$

3. If A, B, C, D · · · etc. be n points in a plane, O their centre of mean position, and P any other point in the plane, then

$$AP^2 + BP^2 + CP^2 + \text{etc.} = AO^2 + BO^2 + CO^2 + \text{etc.} + nOP^2.$$

CHAPTER II

PROPERTIES OF A TRIANGLE

Note.—In this Chapter references such as (III. 22) are to the *Syllabus of Plane Geometry (corresponding to Euclid* I.-VI.) New Edition.

Definitions

(1) Any straight line drawn across a system of lines is called a TRANSVERSAL.

(2) If three or more straight lines intersect in the same point, they are said to be CONCURRENT.

(3) If three or more points lie on the same straight line they are said to be COLLINEAR.

(4) If four or more points lie on the circumference of the same circle, they are said to be CONCYCLIC.

(5) A system of three straight lines which are not concurrent, in which the three points of intersection are taken into account; or a system of three points which are not collinear, in which the three straight lines of connection are taken into account, is called a TRIANGLE.

This definition, it will be observed, differs from that given in elementary plane geometry in that it is "descriptive" and not "metrical"; it is a configuration made by three infinite straight lines, and no reference is made to the lengths of the intercepted segments or to the area contained by them. In this chapter we

shall, however, prove several metrical theorems; and no confusion, it is hoped, will be found in occasionally using the term triangle in the Euclidian sense.

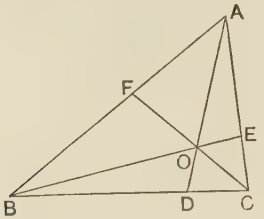
In the following theorems, where the sense as well as the magnitude of the segments of the sides of the triangle ABC is taken into account, the positive directions of measurement will be considered to be from A to B, from B to C, and from C to A. The sides are with this convention said to be taken in *cyclic* order. In the figures drawn, the triangle ABC is acute-angled, AB is the greatest side, and CA the least; and the theorems are enunciated so that the magnitudes of the lines, etc., are positive in these figures. If different figures are drawn, and we recommend the student to draw such, obtuse-angled triangles for instance, it will be found that the theorems are still strictly true, but that the sense of the magnitudes will have to be considered. Thus in Theorem 12, Cor., we prove that $X_1X = c - b$; if b be $> c$, the theorem is still true, or it may be stated $XX_1 = b - c$. Similarly in Theorem 22, Cor., OAH is always $= C - B$, but if B be $> C$ it will be found that AH is on the other side of OA to what it is in the figure.

Theorem 1

If transversals through the angular points A, B, C *of a triangle are concurrent and intersect the opposite sides in* D, E, F *respectively, then*

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$$ (*Ceva's Theorem*).

Let the transversals meet in O; then



$$BD : DC = \Delta ABD : \Delta ADC$$

and $$BD : DC = \Delta OBD : \Delta ODC$$

$$\therefore BD : DC = \Delta ABO : \Delta CAO.$$

Similarly $$CE : EA = \Delta BCO : \Delta ABO,$$

and $$AF : FB = \Delta CAO : \Delta BCO;$$

$$\therefore BD \cdot CE \cdot AF : DC \cdot EA \cdot FB = ABO \cdot BCO \cdot CAO : CAO \cdot ABO \cdot BCO,$$

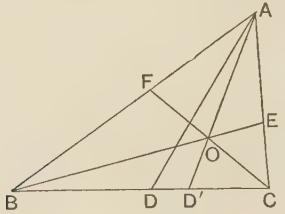
or $$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB.$$

Theorem 2

If D, E, F *be points in the sides* BC, CA, AB *respectively, of a triangle* ABC, *such that*

$$BD \cdot CE \cdot AF = DC \cdot EA \cdot FB,$$

then AD, BE, CF *are concurrent* (*converse of Theorem* 1).



Let BE, CF meet in O, and if AD does not pass through O, draw AO meeting BC in D′; then

$$BD'.CE.AF = D'C.EA.FB,$$

but $$BD.CE.AF = DC.EA.FB;$$

$\therefore BD' : D'C = BD : DC;$

$\therefore$ D′ must coincide with D [Introduction (2)].

Theorem 3

The three internal and three external bisectors of the angles of a triangle meet three by three in four points, the centres of the inscribed and escribed circles.

Let AD, BE, CF be the internal, and AD′, BE′, CF′ the external bisectors; then, since

$$BD : DC = AB : CA \quad (V.\ 9)$$
$$CE : EA = BC : AB$$
$$AF : FB = CA : BC;$$

$\therefore BD.CE.AF : DC.EA.FB =$
$$AB.BC.CA : CA.AB.BC;$$

and $\therefore$ AD, BE, CF are concurrent (Theorem 2).

Similarly it may be proved for AD, BE′, CF′; for BE, CF′, AD′; and for CF, AD′, BE′.

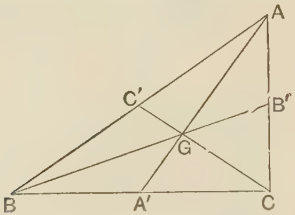
Or the theorem may be proved directly from the problem of describing the circles which touch three given straight lines (III. Prob. 7).

Def.—We shall call the circle inscribed in a triangle the INCIRCLE, and its centre the INCENTRE, and the escribed circles the EXCIRCLES, and their centres the EXCENTRES.

Theorem 4

The straight lines joining the angular points of a triangle to the midpoints of the opposite sides are concurrent.

This follows at once from Theorem 2, or may be proved by elementary geometry thus :—



Let A′, B′, C′ be the midpoints of the sides. Join BB′, CC′ meeting in G, and join AG, GA′.

Then $\Delta\ BB'C = \Delta\ BB'A$,

and $\Delta\ GB'C = \Delta\ GB'A$;

$\therefore\ \Delta\ BGC = \Delta\ BGA$.

Similarly $\Delta\ BGC = \Delta\ CGA$;

$\therefore\ \Delta\ BGA = \Delta\ CGA$;

also $\Delta\ BGA' = \Delta\ CGA'$;

$\therefore$ fig. $BAGA' =$ fig. $CAGA'$;

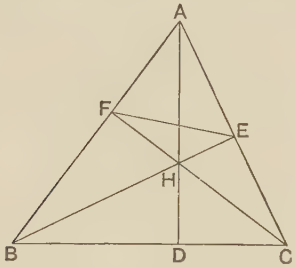
and $\therefore$ each is half of $\Delta\ ABC$, and G must lie on AA′.

DEF.—The three transversals in this theorem are called the MEDIANS, and their point of concurrence the CENTROID of the triangle.

Theorem 5

The perpendiculars from the angular points of a triangle on the opposite sides are concurrent.

Let AD, BE, CF be the three perpendiculars, then, since the triangles BDA, BFC are similar,



$$BD : FB = AB : BC.$$

Similarly $CE : DC = BC : CA,$

and $AF : EA = CA : AB;$

$\therefore BD . CE . AF : FB . DC . EA =$

$$AB . BC . CA : BC . CA . AB;$$

$$\therefore BD . CE . AF = DC . EA . FB,$$

and AD, BE, CF are concurrent (Theorem 2).

This theorem may also be proved independently, thus:—

Let BE, CF, the perpendiculars on CA, AB, cut in H. Join AH and draw HD perpendicular to BC.

Then angle AHF = AEF $\because$ A, E, H, F are concyclic,

= suppt. FEC

= FBC $\because$ E, F, B, C are concyclic,

= suppt. FHD $\because$ F, B, D, H are concyclic;

$\therefore$ AHD is a straight line;

$\therefore$ the perpendiculars are concurrent.

DEF.— The three perpendiculars are called the ALTITUDES, and their point of concurrence the ORTHOCENTRE of the triangle.

Theorem 6

The lines joining the angular points of a triangle to the points where the incircle touches the sides are concurrent.

If X, Y, Z be the points in BC, CA, AB where the circle touches these lines, then, since

$$AZ = YA,\ BX = ZB,\ CY = XC,$$

the theorem follows at once from Theorem 2.

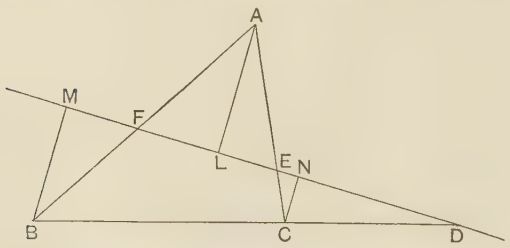
COR.—The theorem may be similarly proved for the excircles.

Theorem 7

If a transversal meet the sides BC, CA, AB *of a triangle in* D, E, F *respectively, then*

$$BD \cdot CE \cdot AF = -DC \cdot EA \cdot FB$$ (*Menelaus' Theorem*).

Draw AL, BM, CN perpendicular to DEF. Then



from similar triangles, having regard to the sense of the segments as given by the figure, we have

$$\begin{aligned} BD : -DC &= BM : CN \\ CE : EA &= CN : -AL \\ AF : FB &= -AL : BM ; \end{aligned}$$

$$\therefore BD \cdot CE \cdot AF : -DC \cdot EA \cdot FB = -BM \cdot CN \cdot AL : -CN \cdot AL \cdot BM ;$$

$$\therefore BD \cdot CE \cdot AF = -DC \cdot EA \cdot FB.$$

Theorem 8

If D, E, F *be points in the sides* BC, CA, AB *respectively of a triangle* ABC, *such that*

$$BD \cdot CE \cdot AF = -DC \cdot EA \cdot FB,$$

then D, E, F *are collinear* (*converse of Theorem* 7).

Suppose that FE meets BC in D′, then by Theorem 7

$$BD' \cdot CE \cdot AF = -D'C \cdot EA \cdot FB,$$

but $$BD \cdot CE \cdot AF = -DC \cdot EA \cdot FB ;$$

$$\therefore BD' : D'C = BD : DC,$$

and ∴ D′ coincides with D [Introduction (2)].

Theorem 9

The three internal and three external bisectors of the angles of a triangle meet the opposite sides in six points, which lie three by three in four straight lines.

Let the three internal bisectors meet the opposite sides in D, E, F, and the external bisectors meet them in D′, E′, F′ ; then

$$\begin{aligned} BD : DC &= AB : CA \\ CE : EA &= BC : AB \\ -AF' : F'B &= CA : BC ; \end{aligned}$$

$\therefore -BD \cdot CE \cdot AF' = DC \cdot EA \cdot F'B$;

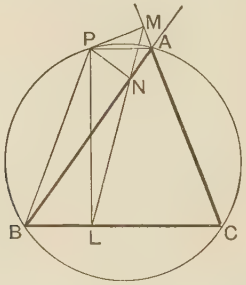
$\therefore$ D, E, F′ are collinear (Theorem 8).

Similarly E, F, D′; F, D, E′; and D′, E′, F′ are collinear.

Theorem 10

The feet of the perpendiculars on the sides of a triangle drawn from any point in the circumference of its circumscribing circle are collinear.

Let ABC be the triangle, and PL, PM, PN the perpendiculars on BC, CA, AB respectively. Join LN, NM.



$\because$ PNB and PLB are right angles,

$\therefore$ P, N, L, B are concyclic.

Similarly P, M, A, N are concyclic;

$\therefore$ angle PNL = suppt. PBL

= PAC

= suppt. PAM

= suppt. PNM;

$\therefore$ LMN is a straight line.

DEFS.—The line LMN is called the SIMSON or PEDAL line of the point P.

The circle which circumscribes a triangle is called the CIRCUMCIRCLE, its centre the CIRCUMCENTRE, and its radius the CIRCUMRADIUS.

In Theorems 11-32 the notation for the triangle ABC will be as follows:—

A′, B′, C′ are the middle points of the sides BC, CA, AB respectively;

AD, BE, CF, the perpendiculars on these sides;

G, H, the centroid and orthocentre respectively;

O, the circumcentre, N, the nine-point centre;

I, I_1, I_2, I_3, the incentre and excentres respectively;

X, Y, Z, the points where the incircle, and

X_1, Y_1, Z_1, the points where the excircle opposite A touches the sides BC, CA, AB respectively;

R, r, r_1, r_2, r_3, the circum-, in-, and exradii respectively;

a, b, c, s, Δ, the lengths of the sides, semiperimeter, and area.

Theorem 11

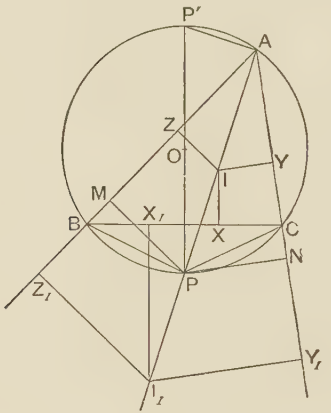
$$AZ + BC = BX + CA = CY + AB = \textit{semiperimeter},$$
$$\textit{or } AZ = YA = s - a, \textit{ etc.}$$

For AZ = YA, BX = ZB, CY = XC (III. 22 Cor.)

and these six lines together make up the perimeter;

$$\therefore AZ + BX + XC = \text{semiperimeter};$$

that is, $AZ + BC =$ semiperimeter, or $AZ = s - a$,
and similarly $BX = s - b$, $CY = s - c$.



Theorem 12

$AZ_1 = BX_1 + AB = X_1C + CA =$ *semiperimeter ;*
or $AZ_1 = Y_1A = s$; $BX_1 = BZ_1 = s - c$; $X_1C = Y_1C = s - b$.

$$\begin{aligned}\text{Perimeter} &= AB + BX_1 + X_1C + CA \\ &= AB + BZ_1 + Y_1C + CA \text{ (III. 22 Cor.)} \\ &= AZ_1 + Y_1A = 2AZ_1 ;\end{aligned}$$

$\therefore AZ_1 = BX_1 + AB = Y_1A = X_1C + CA =$ semiperimeter ;
or $AZ_1 = Y_1A = s$; $BX_1 = BZ_1 = s - c$; $X_1C = Y_1C = s - b$.

Cor.—$X_1X = BX - BX_1 = (s - b) - (s - c) = c - b$.
Also $XX_2 = BX_2 - BX = s - (s - b) = b$.
Similarly $XX_3 = c$.

Theorem 13

$$\Delta = rs = r_1(s-a) = r_2(s-b) = r_3(s-c).$$

For Δ = sum of triangles BCI, CAI, ABI

$$= \tfrac{1}{2}(ra + rb + rc) = rs.$$

Again, $\Delta = \mathrm{ABI}_1 + \mathrm{CAI}_1 - \mathrm{BCI}_1$

$$= \tfrac{1}{2}(r_1 c + r_1 b - r_1 a) = r_1(s-a);$$

and similarly for the rest.

Theorem 14

$$\mathrm{AA'} = 3\mathrm{GA'};\ \mathrm{BB'} = 3\mathrm{GB'};\ \mathrm{CC'} = 3\mathrm{GC'}.$$

Join A'B' (see fig., Theorem 4), which is parallel to AB, and the triangles ABC, B'A'C are similar.

$\therefore$ AB : B'A' = BC : A'C = 2 : 1.

Also ABG and A'B'G are similar;

$\therefore$ AG : GA' = AB : B'A' = 2 : 1;

$\therefore$ AA' = 3GA'; and similarly for the rest.

Theorem 15

If the straight lines which bisect the interior and exterior angles between AB *and* AC *meet the circumcircle in* P, P', *then* PP' *is the diameter of the circle which bisects* BC.

The line which bisects BC at right angles, and the bisector of the angle BAC, both bisect the arc BC; and the former line PA'P' is a diameter of the circumcircle (III. 10).

$\therefore$ the angle PAP′, in a semicircle, is a right angle, and consequently AP′ bisects the exterior angle between AB and AC, which proves the proposition.

Theorem 16

$$PI = PB = PC.$$

Join BI. Then angle $BIP = IAB + IBA = \frac{1}{2}(A + B)$.
Again, $IBP = PBC + IBC = PAC + IBC$ (III. 16)
$= \frac{1}{2}(A + B)$;
$\therefore$ BIP = IBP, and $\therefore$ PI = PB = PC.

Cor.—Since ICI_1 is a right angle, the circle described with centre P at distance PI must pass through B, C, and I_1, and

$$\therefore PI_1 = PI = PB = PC.$$

Theorem 17

If PM *be perpendicular to* AB, *then*

$$2AM = b + c;\ 2MB = c - b.$$

Draw PN perpendicular to AC; then the triangles PAM, PAN are equal in all respects, and

$$\therefore AM = AN,\ PM = PN.$$

Again, in the triangles PBM, PCN, PB = PC, PM = PN, and the angles at M and N are right angles;

$$\therefore MB = NC \text{ (I. 20 Cor. 1).}$$

Now $2AM = AM + AN = AB - MB + AC + CN = c + b$.
Also $2MB = 2(AB - AM) = 2AB - (c + b) = c - b$.

Theorem 18

$$4R\ \Delta = abc.$$

The angle $AOB = 2C$ (III. 15);

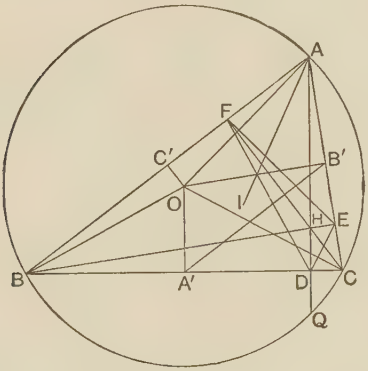
$\therefore$ $BOC' = C$, and

$\therefore$ the triangles ADC, BC'O are similar, and

$$AD : AC = BC' : BO;$$

$\therefore$ $2R \cdot AD = bc$; $\therefore$ $2R \cdot AD \cdot a = abc$,

or $$4R\ \Delta = abc.$$



Theorem 19

Of the four points A, B, C, H, *any one is the orthocentre of the triangle formed by joining the other three.*

It is obvious that the three perpendiculars HD, BF, CE, from the angles of the triangle HBC on the opposite sides meet in A; and so for the other triangles.

Theorem 20

Angle BHF = CHE = A
„ CHD = AHF = B
„ AHE = BHD = C.

BHF is the complement of HBF, and A is the complement of the same angle ;

∴ BHF = CHE = A ; and so for the others.

Theorem 21

The triangles AEF, DBF, DEC *are each similar to* ABC.

Since BEC and CFB are right angles, the semicircle on BC passes through E and F.

∴ FEC + B = 2 right angles = FEC + FEA ;

∴ FEA = B. Similarly EFA = C ;

∴ the triangle AEF is similar to ABC, and similarly for the others.

Theorem 22

AI, BI, CI *bisect the angles* OAH, OBH, OCH *respectively.*

The angle AOC′ is half AOB, and so is equal to C ; and AC′O, ADC are right angles ;

∴ OAC′ = CAD. And since AI bisects BAC, it follows that it must also bisect OAH ; and similarly for BI and CI.

Cor.—$OAH = A - 2DAC = A - 2(90° - C)$
$= A - (A + B + C) + 2C$
$= C - B.$
Similarly $OBH = C - A$, and $OCH = A - B$.

Theorem 23

A, B, C, H *are the centres of the four circles which touch the sides of the orthocentric triangle* DEF.

Since HD is perpendicular to BC, the angles HDE, HDF are respectively the complements of CDE and FDB, and these latter angles are each equal to A (Theorem 21).
$\therefore$ DH and BDC bisect the interior and exterior angles between FD and DE, from which the theorem follows.

Theorem 24

$AH = 2OA'$; $BH = 2OB'$; $CH = 2OC'$.

The sides of the triangle HAB are parallel respectively to those of OA′B′, and consequently the triangles are similar; and as $AB = 2A'B'$, it follows that $AH = 2OA'$, $BH = 2OB'$, and similarly $CH = 2OC'$.

Theorem 25

If AD *be produced to meet the circumcircle in* Q, *then* HD = DQ.

Join BQ; then angle QBC = QAC
= complement of C
= CBE.

And BD is common to the two triangles HBD and QBD, and is at right angles to HQ;

$\therefore$ HD = DQ.

Theorem 26

Rectangle AH . HD = BH . HE = CH . HF.

The triangles AHE, BHD are obviously similar;

$\therefore$ AH : HE = BH : HD;

$\therefore$ rectangle AH . HD = BH . HE = CH . HF (from Δs AHF and CHD).

Theorem 27

A circle passes through A′, B′, C′, D, E, F, *and the mid-points of* HA, HB, HC.

Join B′D, C′D; then since B′ is the centre of the circle described on AC as diameter, B′D = B′A;

$\therefore$ angle B′DA = B′AD.

Similarly C′DA = C′AD;

$\therefore$ C′DB′ = A.

Again, since C′A′ and B′A′ are respectively parallel to CA and BA, the angle C′A′B′ = A;

$\therefore$ C′DB′ = C′A′B′;

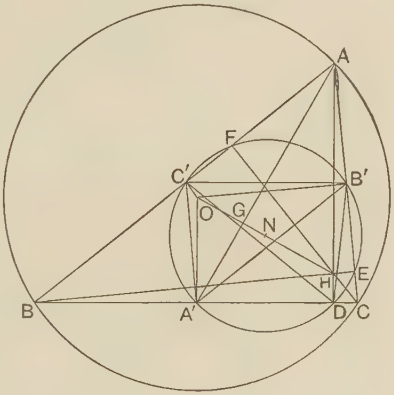
$\therefore$ the circle round A′B′C′ passes through D.

Similarly it passes through E and F. That is, the

circumcircle of DEF passes through the midpoints of the sides of the triangle ABC.

Similarly the same circle passes through the midpoints of the sides of HBC, HCA, HAB (Theorem 19).

Cor.—The line bisecting A′D at right angles passes through the centre of this circle, and also through the midpoint of OH; and as the line bisecting B′E at right angles passes through the same two points, it follows that the centre of this circle is at the midpoint N of OH.



Also, since the sides of the triangle A′B′C′ are halves of the sides of ABC, it follows that the diameter of the circumcircle of A′B′C′ is half of the diameter of the circumcircle of ABC, and is $\therefore = R$.

Def.—The circle treated of in this theorem is called the NINE-POINT CIRCLE of the triangle ABC.

Theorem 28

G *lies on* OH, *and* OH = 3OG.

Let AA′ meet OH in G′. Then the triangles AG′H, A′G′O are similar, and as AH = 2A′O (Theorem 24);

∴ G′H = 2OG′, and AG′ = 2G′A′.

But AG = 2GA′;

∴ G′ coincides with G, and GH = 2OG;

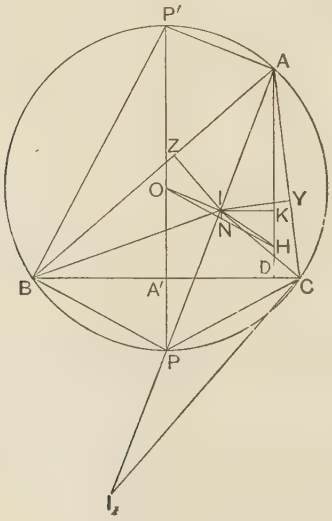
that is, OH = 3OG.

Theorem 29

$$IO^2 = R^2 - 2Rr.$$

Join P′B. Then, since angle BP′P = BAP, the triangles BP′P and ZAI are similar;

∴ PP′ : PB = AI : IZ,



or $$2Rr = PB \cdot AI = PI \cdot AI \text{ (Theorem 16)}$$
$$= R^2 - OI^2 \text{ (III. 28)};$$
$$\therefore OI^2 = R^2 - 2Rr.$$

Theorem 30

$$IH^2 = 2r^2 - AH \cdot HD.$$

Join CI_1, and draw IK perpendicular to AD.
Then ICI_1 is a right angle $= \frac{1}{2}(A + B + C)$;

$$\therefore ACI_1 = \frac{A}{2} + \frac{B}{2} + C = AIB;$$

$\therefore$ the triangles AIB, ACI_1 are similar, and
$\therefore AI \cdot AI_1 = AB \cdot AC = 2R \cdot AD$ (Theorem 18);
that is, $2AI \cdot AP - AI^2 = 2R \cdot AD$ (since $PI = PI_1$, Theorem 16, Cor.);
$\therefore 2AK \cdot 2R - AI^2 = 2R \cdot AD$ (similar Δs AIK, PP′A),
or $4R \cdot AK = 2R \cdot AD + AI^2$.

Again, $IH^2 = AI^2 + AH^2 - 2AH \cdot AK$,
or $2R \cdot IH^2 = 2R \cdot AI^2 + 2R \cdot AH^2 - AH \cdot 4R \cdot AK$
$= 2R \cdot AI^2 + 2R \cdot AH^2 - AH(2R \cdot AD + AI^2)$
$= (2R - AH)AI^2 - 2R \cdot AH(AD - AH)$
$= (2OP - 2OA')AI^2 - 2R \cdot AH \cdot HD$ (Theorem 24)
$= 2A'P \cdot AI^2 - 2R \cdot AH \cdot HD$
$= 2r \cdot PB \cdot AI - 2R \cdot AH \cdot HD$ (similar Δs BA′P, AYI)
$= 2r \cdot r \cdot 2R - 2R \cdot AH \cdot HD$ (similar Δs P′BP, AYI);
$\therefore IH^2 = 2r^2 - AH \cdot HD.$

Theorem 31

$$OH^2 = R^2 - 2AH \cdot HD.$$

For $AH \cdot HQ = R^2 - OH^2$ (III. 28);

$\therefore\ AH \cdot 2HD = R^2 - OH^2$ (Theorem 25),

or $$OH^2 = R^2 - 2AH \cdot HD.$$

Theorem 32

$$NI = \tfrac{1}{2}R - r.$$

Since N is the midpoint of OH,

$$2IN^2 + \tfrac{1}{2}OH^2 = IO^2 + IH^2 \text{ (II. 12)};$$

that is, $2IN^2 + \frac{1}{2}R^2 - AH \cdot HD = R^2 - 2Rr + 2r^2$

$- AH \cdot HD$ (Theorems 29, 30, 31);

$$\therefore\ 4IN^2 = R^2 - 4Rr + 4r^2,$$

or $$2IN = R - 2r.$$

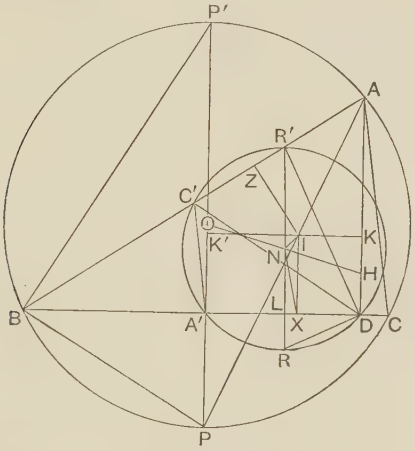
Cor.—The theorem proves that the difference of the radii of the nine-point circle and the incircle is equal to the distance between their centres, and consequently that the circles touch each other internally. This property is usually called Feuerbach's Theorem.

The following independent proof of this interesting theorem is a modification of one by Mr. W. Harvey, extracted from the Proceedings of the Edinburgh Mathematical Society, vol. v. p. 102.

Draw the diameter RLNR′ of nine-point circle which bisects A′D at right angles.[1] Draw KIK′ parallel to A′D. Join RD, R′D, C′D, C′A′, NX.

[1] The figure on p. 33 has been unintentionally drawn so that R′ falls on AB; this need not be so, and for the purposes of the proposition it is immaterial whether R′ is on AB or not.

Then, since C′ is the circumcentre of ABD, AC′D = 2B; and since C′A′ is parallel to CA, AC′A′ is supplement of A, and ∴ = B + C;



∴ A′C′D C − B. And RR′D = ½A′C′D =
½(C − B) = IAD (Theorem 22, Cor.);
∴ the triangles RR′D, RDL, IAK, IPK′ are all similar.

Hence $\frac{A'X}{BP} = \frac{K'I}{PI} = \frac{RD}{R}$ (Theorems 16 and 27, Cor.)

and $\frac{XD}{AI} = \frac{IK}{AI} = \frac{RL}{RD}$;

also $\frac{BP}{2R} : \frac{IZ}{AI} = \frac{r}{AI}$ (similar △s PBP′, IZA);

$$\therefore \frac{A'X}{BP} \cdot \frac{XD}{AI} \cdot \frac{BP}{2R} = \frac{RD}{R} \cdot \frac{RL}{RD} \cdot \frac{r}{AI};$$

that is, $$A'X \,.\, XD = 2r \,.\, RL,$$

or $$\frac{R^2}{4} - NX^2 = 2r \cdot RL \text{ (III. 28) } \ldots \text{ (1).}$$

Again, from Δ NIX, $NX^2 + IX^2 - NI^2 = 2IX \cdot NL$
$= 2r \cdot NL$;

$\therefore$ from (1) $\frac{R^2}{4} + r^2 - NI^2 = 2r \cdot NR = rR$;

$$\therefore NI^2 = \left(\frac{R}{2} - r\right)^2.$$

Hence the proposition follows as before.

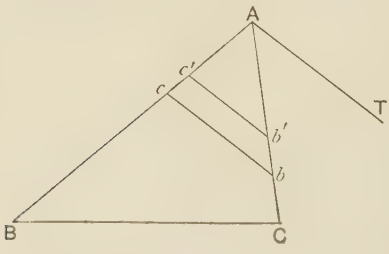
Another proof of this theorem depending on the principle of inversion will be given later on.

It may be remarked that by similar proofs to the above it may be demonstrated that the nine-point circle touches the escribed circles. For the escribed circle opposite A, theorems 29 and 30 become

$$I_1O^2 = R^2 + 2Rr_1, \quad I_1H^2 = 2r_1^2 - AH \cdot HD.$$

ANTIPARALLELS, SYMMEDIAN LINES AND POINTS

DEF. —If ABC be a triangle, and any points b, c, be taken on CA, AB respectively, in such a manner that the angles Abc, ABC are equal, then will also the angles Acb, ACB be equal, and the triangles Abc, ABC similar; and with reference to AB, AC, the base bc is said to be ANTIPARALLEL to the base BC.



NOTES.—The points B, c, b, C are concyclic, since the opposite angles of the quadrilateral BcbC are equal to two right angles.

The antiparallel bc is fixed as regards direction, being obviously parallel to any other antiparallel $b'c'$, and being parallel to the tangent AT to the circumcircle at A; for the angle TAC = ABC in the alternate segment = Abc.

Theorem 33

If PQR *be the triangle formed by tangents to the circumcircle at* A, B, C, *then* AP, BQ, CR *bisect all the antiparallels to* BC, CA, AB *respectively, and are concurrent.*

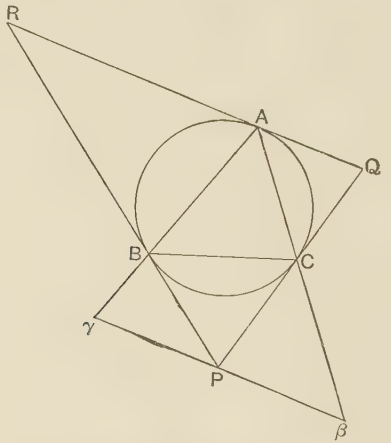
Through P draw to BC the antiparallel $\beta P\gamma$ (see fig., p. 36), which is consequently parallel to QAR.

Then angle $P\gamma B = RAB$
$= RBA$ (since $RA = RB$)
$= \gamma BP$;
$\therefore P\gamma = PB$.
Similarly $P\beta = PC$.

And since $PB = PC$, P is the midpoint of $\beta\gamma$. And as $\beta\gamma$ is bisected by AP, all lines parallel to $\beta\gamma$ and terminated by AB, AC—that is, all antiparallels to BC are bisected by AP; and similarly for BQ and CR.

Also, since $BP = PC$, $CQ = QA$, $AR = RB$,
$\therefore PC \cdot QA \cdot RB = CQ \cdot AR \cdot BP$;
$\therefore$ PA, QB, RC are concurrent (Theorem 2).

DEF.—A line drawn through the vertex of a triangle so as to bisect the antiparallels to the base is called a



SYMMEDIAN. The point of concurrence K of the three symmedian lines AP, BQ, CR is called the SYMMEDIAN POINT (*point de Lemoine*).

NOTE.—The term "symmedian" comes from the analogous properties of the medians of a triangle; that as the medians AA′, BB′, CC′ bisect all parallels to BC, CA, AB respectively which are terminated by the sides, and concur at G the centroid (Theorem 4), so the symmedian lines AP, BQ, CR bisect the antiparallels and concur at K the symmedian point.

Theorem 34

The lines joining K, G *to any vertex of the triangle are equally inclined to the bisector of the angle.*

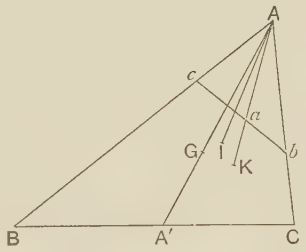
Draw any antiparallel bc to BC, and let it be bisected by AK at a.

Then, since AB : BC = Ab : bc;

$\therefore$ AB : BA′ = Ab : ba;

$\therefore$ the triangles ABA′, Aba are similar, and angle BAA′ = bAa.

Similarly CAA′ = cAa. And since BAI = CAI, it follows that GAI = KAI.



Def.—Pairs of points possessing the property proved in this theorem of K and G are called ISOGONAL CONJUGATES.

Theorem 35

If S, S′ *are two points which are isogonal with respect to the angles* A *and* B *of the triangle* ABC, *then they are also isogonal with respect to* C; *and the rectangles under their perpendiculars on the three sides are equal.*

Let L, M, N and L′, M′, N′ be the feet of the perpendiculars from S, S′ on the sides.

By similar triangles SN : SA = S′M′ : S′A

SA : SM = S′A : S′N′;

$\therefore$ SN : SM = S′M′ : S′N′,

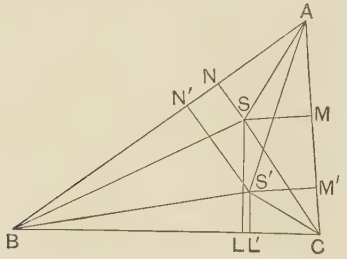
or SN . S′N′ = SM . S′M′.

Similarly SN . S′N′ = SL . S′L′;

$\therefore$ SM . S′M′ = SL . S′L′,

or SL : SM = S′M′ : S′L′.

That is, CS, CS′ divide the angle ACB, so that perpendiculars from points in them on the sides have the same ratio, and ∴ isogonally.



Theorem 36

The symmedian point K *has its perpendicular distances from the sides respectively proportional to those sides.*

Since AA′ = 3GA′ (Theorem 14), it follows that the perpendiculars from the centroid G on the sides are respectively one-third of AD, BE, CF; and as AD . BC = BE . CA = CF . AB = 2Δ, the perpendiculars from G on the sides are inversely proportional to the sides, and ∴ the perpendiculars from K are proportional to the sides.

Theorem 37

The sides of the orthocentric triangle are antiparallel to the sides of the original triangle, and are respectively perpendicular to the corresponding radii of the circumcircle.

B, F, E, C are obviously concyclic, so that EF is anti-parallel to BC. EF is ∴ parallel to the tangent at A to the circumcircle, and consequently perpendicular to the radius OA.

BROCARD POINTS

Theorem 38

If a circle be described touching AB *in* A *and passing through* C, *and* AR *be the chord parallel to* BC, *then* BR *will meet the circle again in a point* Ω, *such that the angle* ΩBC = ΩCA = ΩAB.

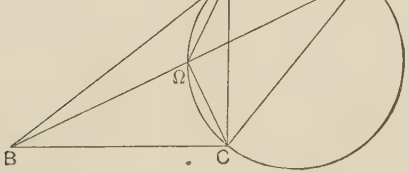
Since AR is parallel to BC,

angle ΩBC = ΩRA

= ΩCA (in the same segment).

Also, since AB touches the circle, ΩAB = ΩCA.

Def.—Each of these equal angles is called the Brocard angle of the triangle, and is denoted by ω.



Theorem 39

The Brocard angle of a triangle depends only upon the

shape of the triangle, and is therefore the same for all similar triangles.

Join CR (last fig.) Then angle BAC = CRA in the alternate segment; and since AR is parallel to BC, BCA = CAR; ∴ the triangles ABC and RCA are similar; and consequently for all triangles similar to ABC, the quadrilaterals formed as ABCR is will be similar, and the angle RBC, that is, ω will be the same for all.

DEF.—The point Ω is called the POSITIVE BROCARD POINT. A NEGATIVE BROCARD POINT exists, being determined by describing a circle to touch AC in A, and passing through B, drawing the chord AR′ parallel to BC, and joining CR′, which meets the circle in Ω'.

In this case angle $\Omega'CB = \Omega'AC = \Omega'BA = \omega$; for it can easily be shown that the triangles R′CB, RBC are identically equal, and ∴ $\Omega'CB = \Omega BC = \omega$. Hence the points Ω, Ω' are isogonal conjugates, and the rectangles under their perpendiculars upon any side are the same.

Theorem 40

The angle $B\Omega C$ = *supplement of* $C = C\Omega'A$

,, $C\Omega A$ = ,, $A = A\Omega'B$

,, $A\Omega B$ = ,, $B = B\Omega'C$.

For $B\Omega C$ is the supplement of $\Omega BC + \Omega CB$

,, ,, ,, $\omega + (C - \omega)$

,, ,, ,, C.

And similarly for the others.

NOTE.—The two Brocard points may be distinguished

in the following manner. For the positive point we proceed in measuring ω in the positive cyclic order, ΩBC, ΩCA, ΩAB; whilst for the negative point the order is reversed, $\Omega' CB$, $\Omega' BA$, $\Omega' AC$.

In the following theorems the triangle ABC is supposed to be lettered in such a manner that A is the upper vertex, B the left, and C the right extremity of the base.

TRIPLICATE RATIO CIRCLE

Theorem 41

If through the symmedian point K *parallels* F'KE, D'KF, E'KD *be drawn to the sides, the six points* D, D', E, E', F, F' *all lie on a circle, whose centre* σ *is the midpoint of* KO (see fig., p. 42).

Since AFKE' is a parallelogram, AK bisects FE', and

$\therefore$ FE' is antiparallel to BC or EF';

$\therefore$ E, E', F, F' are concyclic.

Similarly DF', AC are antiparallel;

$\therefore$ angle BF'D = C;

$\therefore$ E'DF' = E'EF';

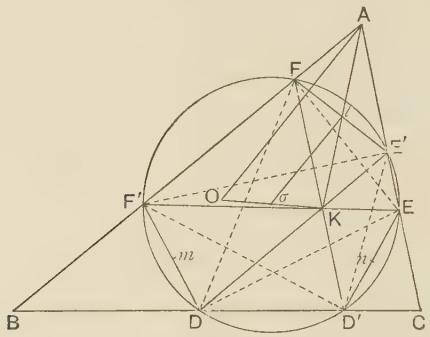
$\therefore$ the same circle passes through D; and similarly it passes through D'.

Again, let AK, FE' intersect in l, join $l\sigma$, AO, which are parallel, and since AO is perpendicular to FE' (Theorem 37),

$\therefore$ $l\sigma$ is perpendicular to FE', and also bisects it.

Similarly the lines joining the midpoints m, n of DF' and ED' to σ are perpendicular respectively to these

lines, and so the centre of the circle through F, E′, E, D′, D, F′ lies in $l\sigma$, $m\sigma$, $n\sigma$; that is, σ is the centre.



Theorem 42

$$DD' : EE' : FF' = a^3 : b^3 : c^3 \text{ (}\textit{see last fig.}\text{)}$$

From similar triangles $DD' : D'K = a : b$

$$CE : D'C = a : b$$

$$KE : EE' = a : b.$$

Compounding these ratios and observing that $CE = D'K$ and $KE = D'C$, we get $DD' : EE' = a^3 : b^3$.

Def.—From this theorem the circle through D, D′, E, E′, F, F′ is called the TRIPLICATE RATIO CIRCLE.

Theorem 43

The triangles FDE, E′F′D′ *of Theorem* 41 *are equal to each other, and are each similar to* ABC.

For angle FDE = FF′E (in the same segment)
= B,
and similarly FED = FD′D = C.

Hence FDE, ABC are similar.

Similarly E′F′D′, ABC are similar.

And since the triangles FDE, E′F′D′ are equiangular and inscribed in the same circle, they must be equal.

Theorem 44

If Ω, Ω′ *be the Brocard points of* ABC, *then* Ω, K *and* K, Ω′ *are the Brocard points of* FDE *and* E′F′D′ *respectively.*

Since EF′, DD′ are parallel chords of the triplicate ratio circle,

∴ D′E = F′D, and similarly each = E′F.

And as these equal chords subtend equal angles at the circumference of this circle,

∴ the angle KED = KFE = KDF ;
also KD′E′ = KE′F′ = KF′D′ ;

∴ K is the negative Brocard point of FDE, and the positive Brocard point of E′F′D′.

Again, let Π be the positive Brocard point of FDE, then the angle FΠD is the supplement of FDE, that is of B.

Hence B, F, Π, D are concyclic, and

∴ angle ΠFD = ΠBC.

But ΠFD is the Brocard angle of FDE, and is equal

to the Brocard angle ΩBC of the similar triangle ABC (Theorem 39);

∴ BΠ passes through Ω. In like manner AΠ and CΠ pass through Ω;

∴ Ω is the positive Brocard point of FDE.

Similarly it may be shown that Ω′ is the negative Brocard point of E′F′D′.

BROCARD CIRCLE

DEF.—The circle described upon OK as diameter, and which has consequently the same centre as the T. R. circle, is termed the BROCARD CIRCLE.

Theorem 45

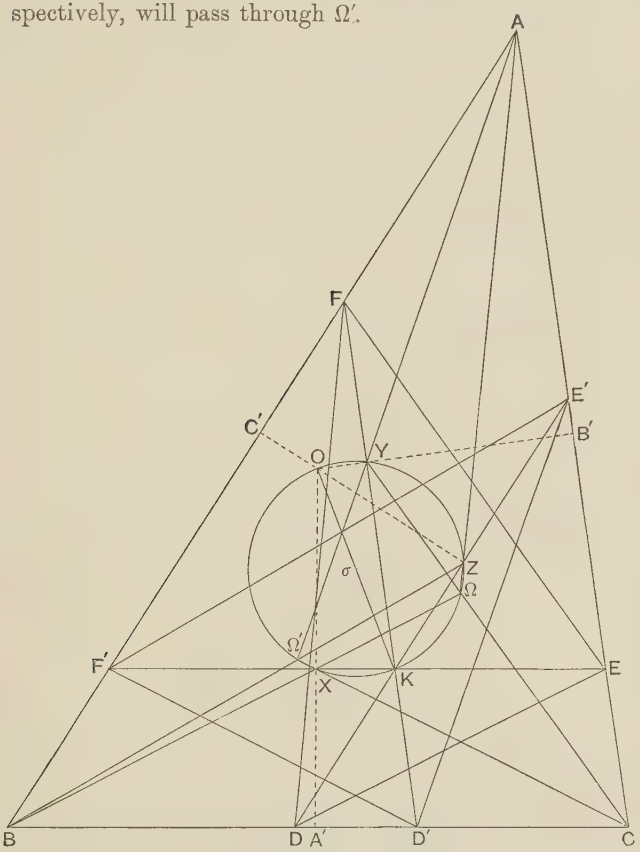
The Brocard circle passes through the Brocard points Ω, Ω′, *and the three points of intersection of* AΩ, BΩ, CΩ, *with* BΩ′, CΩ′, AΩ′ *respectively; and the angle* $\Omega\sigma K = \Omega'\sigma K = 2\omega$.

With the same construction as for the T. R. circle, join AΩ, BΩ, CΩ.

Then angle DFF′ = DE′F′ in same segment of T. R. circle
= ω, since K is the Brocard point of D′E′F′
= ΩAB;

∴ AΩ is parallel to FD; similarly BΩ, CΩ are parallel to DE, EF respectively, and in like manner

the parallels to D′E′, E′F′, F′D′ through A, B, C respectively, will pass through Ω′.



Let BΩ meet F′E in X. Join CX. Then, by parallels, F′K = BD = XE = XK + KE;

$\therefore$ F′X = KE = D′C, and F′X and D′C are parallel;
$\therefore$ CX is parallel to D′F′;
$\therefore$ CX passes through Ω′.

Similarly CΩ, AΩ′ meet in Y on D′F,
and AΩ, BΩ′ meet in Z on E′D.
Then, since angle XBC = ω = XCB, $\therefore$ XB = XC;

$\therefore$ XA′, the perpendicular to BC, bisects BC and also passes through O;
$\therefore$ OXK is a right angle;
$\therefore$ the Brocard circle passes through X, and in like manner through Y and Z.

Again, angle YΩZ = suppt. CΩA = A
= suppt. B′OC′ = YOZ;

$\therefore$ Ω is on the Brocard circle, and in like manner Ω′ also.

Lastly, since σ is the centre of the circle,

angle ΩσK = 2ΩZK = 2ΩAB = 2ω,
and Ω′σK = 2Ω′ZK = 2Ω′BA = 2ω.

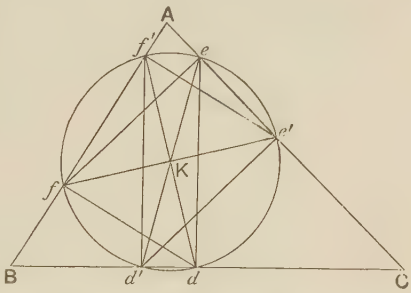
COSINE CIRCLE

Theorem 46

If through the symmedian point of a triangle antiparallels to the sides be drawn, they meet the sides in six points, which lie on a circle called the COSINE CIRCLE *of the triangle.*

Let fKe', dKf', eKd' be the antiparallels; since K is the symmedian point, it is the middle point of the antiparallels; and angle Kdd' $= A = Kd'd$;

$\therefore$ $Kd = Kd' = Ke = Ke' = Kf = Kf'$ in like manner;



$\therefore$ a circle with centre K will pass through d, d', e, e', f, f'.

Cor.—It is obvious that any such figure as $efd'e'$ is a rectangle. Also, since angle $dfe = dd'e$ in same segment $= A$,

and similarly $def = C$, and $edf = B$, the triangle fde is similar to ABC; in like manner $e'f'd'$ is similar to ABC; and since fde and $e'f'd'$ are similar and have the same circumcircle, they must be identically equal.

TAYLOR CIRCLE

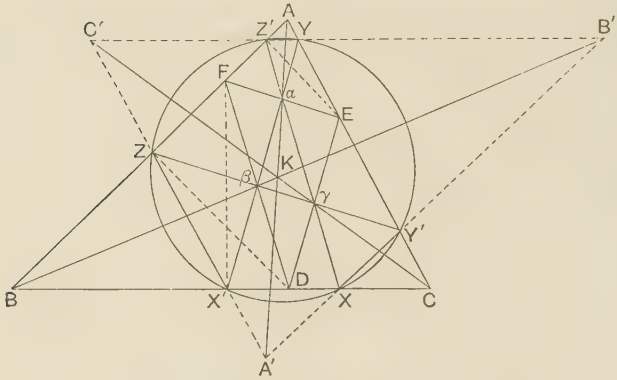
Theorem 47

If DEF *be the orthocentric triangle of the triangle* ABC, *and* α, β, γ *the middle points of its sides, then if*

$\beta\gamma$ *meet* CA, AB *in* Y′, Z *respectively,*
$\gamma\alpha$,, AB, BC ,, Z′, X ,,
$\alpha\beta$,, BC, CA ,, X′, Y ,,

the six points X, X′, Y, Y′, Z, Z′ *will lie on a circle called the* TAYLOR CIRCLE, *whose centre coincides with the incentre of the triangle* $\alpha\beta\gamma$.

Since angle DFZ = C = Y′ZA by antiparallels,



∴ $\beta Z = \beta F$, and $\beta X' = \beta D$ in like manner, but $\beta F = \beta D$;

∴ the circle on FD as diameter passes through Z and X′;

∴ ZX′ is antiparallel to DF or XZ′, and parallel to CA;

∴ Z, Z′, X, X′ are concyclic;

similarly Z′Y is parallel to BC;

$$\therefore\ Z'YX' = YX'C = A = Z'XX';$$

∴ Y, and similarly Y′, lies on the circle through Z, Z′, X, X′.

Also, since $\beta Z = \beta X'$, the perpendicular to ZX′

through its middle point bisects the angle β, and $\therefore$ passes through the incentre of the triangle $\alpha\beta\gamma$. Similarly the perpendiculars to XY', YZ' through their middle points pass through the same point, hence the centre of the circle XX'YY'ZZ' coincides with the incentre of the triangle $\alpha\beta\gamma$.

Cor. 1. The points Z, Y'; X, Z'; Y, X' are the projections on the sides of the original triangle of the feet of the altitudes D, E, F respectively.

Cor. 2. The triangles ZXY, Y'Z'X' are equal to one another and similar to ABC. Proof as in Theorem 43.

TUCKER CIRCLES

Theorem 48

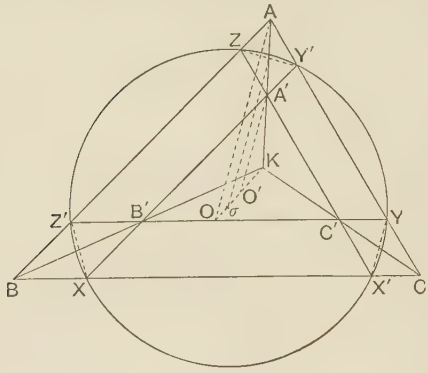
If K *be the symmedian point of the triangle* ABC, *and points* A', B', C' *are taken on* KA, KB, KC, *so that*

$$KA' : KA = KB' : KB = KC' : KC = \text{a constant ratio};$$

then, if B'C' *meet* CA, AB *in* Y, Z' *respectively,*
" C'A' " AB, BC " Z, X' "
" A'B' " BC, CA " X, Y' "

the points X, X', Y, Y', Z, Z' *lie on a circle whose centre bisects the line joining the circumcentres of* ABC, A'B'C'.

Since AZA′Y′ is a parallelogram, ∴ ZY′ and AA′ bisect each other; but AA′ is a symmedian, ∴ ZY′ is antiparallel to BC, and parallel to the tangent to the circumcircle at A; but the triangles ABC, A′B′C′ are similar and similarly situated with respect to K, so the tangents to the circumcircles at A and A′ must be parallel; hence if O and O′ are the circumcentres, the line perpendicular to ZY′ through its middle point is parallel to AO, A′O′, and passes through σ, the middle point of OO′. Similarly the perpendiculars to XZ′, YX′ through their middle points pass through the same point σ.



Also, since ZY′ is antiparallel to BC or Z′Y,

∴ Z, Z′, Y, Y′ are concyclic.

Similarly Z′X is antiparallel to CA, and

∴ angle $Z'XY' = XZ'B = C = Z'YY'$;

∴ X, and similarly X′, lies on the circle through Z, Z′, Y, Y′, and its centre is σ, the middle point of OO′.

Cor.—Since the triangles OAK, O′A′K are similar, O, O′, K are collinear.

DEF.—The circles passing through six points determined as in the above theorem are termed TUCKER CIRCLES.

Theorem 49

The triangles ZXY, Y′Z′X′ *of Theorem* 48 *are equal to one another, and are each similar to* ABC.

The proof is identical with that of Theorem 43.

Theorem 50

The circum-, triplicate ratio, cosine, and Taylor circles are each particular cases of Tucker circles.

As the ratio KA′ : KA changes, the locus of the centre of the Tucker circle is the straight line OK.

When KA′ : KA = 1, A′, B′, C′ coincide with A, B, C respectively, and the Tucker circle becomes the circumcircle.

When KA′ : KA = 0, A′, B′, C′ all coincide with K, and the Tucker circle becomes the triplicate ratio circle.

When KA′ : KA = − 1, the Tucker circle becomes the cosine circle, for the middle point of AA′ will coincide with K, and the antiparallels ZY′, YX′, XZ′ will all pass through K.

In the case of the Taylor circle the ratio KA′ : KA has not a numerical value, but it may be shown by trigonometrical investigation to be equal to

$$1 - \frac{a^2 + b^2 + c^2}{4R^2}, \quad \text{or} \quad \frac{OH^2}{4R^2} - \frac{5}{4}.$$

We can see that the Taylor circle is a Tucker circle, for in the figure of Theorem 47 it was proved that YZ′, ZX′, XY′ are parallel to BC, CA, AB; and if they are produced to meet in A′, B′, C′, then BB′ is a diagonal of the parallelogram BXB′Z′, and therefore bisects Z′X, which is antiparallel to CA; ∴ BB′ is a symmedian, and passes through K; similarly AA′ and CC′ pass through K, and by parallels

$$KA' : KA = KB' : KB = KC' : KC.$$

Examples on Chapter II

1. If the perpendiculars to the sides BC, CA, AB at the points P, Q, R, in these sides respectively, are concurrent, then

$$BP^2 - PC^2 + CQ^2 - QA^2 + AR^2 - RB^2 = 0,$$

and conversely, if this relation exists the perpendiculars are concurrent.

2. When three circles touch, two and two, the three tangents at the points of contact are concurrent.

3. If the excircles touch the sides to which they correspond in X_1, Y_2, Z_3, the perpendiculars to the sides through these points are concurrent.

4. If a circle cut the sides BC, CA, AB in P, P′; Q, Q′; R, R′ respectively, and AP, BQ, CR are concurrent, then AP′, BQ′, CR′ are concurrent.

5. If O be any point, then the external bisectors of the angles BOC, COA, AOB meet the sides BC, CA, AB respectively in three collinear points.

6. If X, Y, Z are the points of contact of the incircle with

the sides, and XI, YI, ZI meet YZ, ZX, XY respectively in L, M, N, then AL, BM, CN meet in the centroid.

7. If XX′, YY′, ZZ′ are diameters of the incircle through X, Y, Z, its points of contact with the sides, then AX′, BY′, CZ′ are concurrent.

8. Tangents to the circumcircle at A, B, C meet BC, CA, AB respectively in three collinear points.

9. Equilateral triangles A_1BC, B_1CA, C_1AB are described external to the triangle ABC, then AA_1, BB_1, CC_1 are equal and concurrent.

10. If BC be divided at K, so that $BK : KC = l : m$, then

$$m \cdot AB^2 + l \cdot CA^2 = (l+m)AK^2 + m \cdot BK^2 + l \cdot KC^2.$$

11. If K, L, M be points in BC, CA, AB respectively, such that $BK : KC = l : m$; $CL : LA = m : k$; $AM : MB = k : l$; and consequently AK, BL, CM meet in S, then

$$SK : AK = lm : lm + mk + kl ;$$

also

$$\frac{SK}{AK} + \frac{SL}{BL} + \frac{SM}{CM} = 1.$$

12. The circumcircle bisects the joins of the in- and ex-centres.

13. The circumcircles of HBC, HCA, HAB are all equal.

14. The circumcircle of IBC intersects AB, CA in points whose distances from A are equal to CA, AB respectively. Hence show that $AI \cdot AI_1 = bc$.

15. If the perpendiculars from any point on the circumcircle to the sides of the triangle meet the circle again in Q, R, S respectively, then AQ, BR, CS are parallel to Simson's line of that point.

16. Simson's line of a point P bisects PH.

17. Find the position of P_1 so that Simson's line may be parallel to a given line.

18. The Simson's lines of the ends of any diameter of the circumcircle intersect at right angles on the nine-point circle.

19. If A'B' meets AI in S, and AI_2 in T, then $A'S = \frac{1}{2}(c-b)$, $A'T = \frac{1}{2}(b+c)$, and ASCT is a rectangle.

20. If p_1, p_2, p_3 be the three altitudes AD, BE, CF respectively, then

$$\frac{2}{p_1} = \frac{1}{r} - \frac{1}{r_1}, \quad \frac{2}{p_2} = \frac{1}{r} - \frac{1}{r_2}, \quad \frac{2}{p_3} = \frac{1}{r} - \frac{1}{r_3},$$

and

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r}.$$

21. If A' is the middle point of BC, and PA'P' the diameter of the circumcircle, $2A'P = r_1 - r$, and $2P'A' = r_2 + r_3$.

22. $r_1 + r_2 + r_3 - r = 4R$.

23. $AH + BH + CH = 2(OA' + OB' + OC') = 2(R + r)$.

24. $r \cdot r_1 = (s-b)(s-c)$. Hence show that

$$\Delta^2 = s(s-a)(s-b)(s-c) = rr_1r_2r_3.$$

25. If A'I be produced to meet AD in L, then $AL = r$; and, generally, the line joining an excentre with the mid-point of any side will meet the corresponding altitude in a point whose distance from the angular point of the triangle is equal to the exradius.

26. If C be a right angle, then

$$r = s - c, \; r_1 = s - b, \; r_2 = s - a, \; r_3 = s.$$

27. $\dfrac{AD^2}{BE \cdot CF} = \dfrac{bc}{a^2}$.

28. $2(AD.AH + BE.BH + CF.CH) = a^2 + b^2 + c^2$.

29. $AH^2 + BC^2 = BH^2 + CA^2 = CH^2 + AB^2$.

30. A′ is the middle point of XX_1.

31. If $r_2 = r_3$, then $b = c$.

32. If $R = 2r$, the triangle is equilateral.

33. A_1, B_1, C_1 are circumcentres of the triangles BHC, CHA, AHB. Show that O is the orthocentre of $A_1B_1C_1$ and H its circumcentre; that A, B, C are the circumcentres of B_1OC_1, C_1OA_1, A_1OB_1, and that all these seven triangles have the same nine-point circle as the triangle ABC.

34. Show that $II_1^2 + I_2I_3^2 =$ square on diameter of circle II_1I_2; and similar relations got by taking I, I_1, I_2, I_3 in any order.

35. The area of the triangle formed by any three of the points I, I_1, I_2, I_3 multiplied by the radius of the circle whose centre is the fourth point $= \frac{1}{2}abc$.

36. Area $I_1I_2I_3 : \Delta = 2R : r$.

37. The circumradius of the orthocentric triangle $= \frac{1}{2}R$.

38. The area of a triangle is a mean proportional between the squares on its circumradius and the semi-perimeter of the orthocentric triangle.

39. Show that I is the orthocentre of the triangle formed by the circumcentres of BIC, CIA, AIB.

40. Show that if BM and CN are perpendicular to AI, then $BM.CN = r.r_1$.

41. Find a point P within a triangle ABC, so that the areas PBC, PCA, PAB may be equal.

42. Find a point P within a given acute-angled triangle ABC such that the circles PBC, PCA, PAB may be equal; and prove that they are also equal to the circle ABC.

43. Find a point within a triangle such that the perpendiculars from it to the sides may be proportional to them.

44. Show that the line joining A′ to the middle point of EF is perpendicular to EF.

45. If DKL is at right angles to DE, cutting BE in K and BA in L, then $LB^2 = LK \cdot LD$.

46. AO meets the circumcircle in L, and AI meets it in P; show that one angle between BL and OP is equal to B.

47. If AA′ meet the circumcircle in K, then KB : KC = AC : AB, and $AC^2 + AB^2 = 2AA' \cdot AK$.

48. If AI meet the circumcircle in P, and PM be drawn perpendicular to AB, then the square on the tangent from M to the circumcircle = CA′ . A′D.

49. If P be any point in the plane of the triangle ABC, show that

$$AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2.$$

50. If N be the nine-point centre,

$$AN^2 + BN^2 + CN^2 = 3R^2 - 9NG^2.$$

51. If K be the middle point of GH,

$$AK^2 + BK^2 + CK^2 = 3R^2.$$

52. If P be any point on the nine-point circle,

$$AP^2 + BP^2 + CP^2 + HP^2 \text{ is constant.}$$

53. Lines are drawn through the midpoints of the sides of a triangle perpendicular to the bisectors of the opposite angles. Prove that the triangle formed by these lines has the same nine-point circle as the original triangle.

54. Three points A′, B′, C′ are taken on the sides BC, CA, AB of a triangle such that BA′ : A′C = CB′ : B′A = AC′ : C′B; show that the centroid of A′B′C′ coincides with that of ABC.

55. Show how to construct a triangle, having given (i) two sides and a median, (ii) one side and two medians, (iii) the three medians.

56. If with the three medians of a triangle a new triangle be formed, the lengths of its medians will be three-fourths of the sides of the original triangle.

57. Given the position (but not the magnitude) of one side, the incentre and the excentre corresponding to the given side, construct the triangle.

58. Show how to construct a triangle, having given—(i) the centres of three of the circles which touch the sides ; (ii) the centres of two of the circles which touch the sides, and the circumcentre ; (iii) the centre of one of the circles which touch the sides, the circumcentre, and an angular point ; (iv) the centroid, the circumcentre, and an angular point.

59. Given the base of a triangle in magnitude and position, and the vertical angle in magnitude, find the locus of (i) the circumcentre ; (ii) the incentre ; (iii) the orthocentre ; (iv) the nine-point centre.

60. Given the base, vertical angle, and (i) the sum, (ii) the difference, (iii) the rectangle of the other two sides, construct the triangle.

61. Given the base, the altitude, and the sum or difference of the sides, construct the triangle.

62. Given the base, the circumradius, and the difference of the angles at the base, construct the triangle.

63. Given the base, the difference of the angles at the base, and the difference of the sides, construct the triangle.

64. Given the circumcircle, the incentre, and one angular point, construct the triangle.

65. Given the nine-point circle and one side, construct the triangle.

66. Given the orthocentre, circumcircle, and one angular point, construct the triangle.

67. Given in magnitude and position the lines joining a vertex of a triangle to the points of contact of the opposite side with the incircle and the excircle corresponding to that side, construct the triangle.

68. Given the circumcircle and orthocentre of a triangle, prove that the lines joining the feet of the altitudes touch a fixed circle.

69. A circle is described concentric with the circumcircle of the triangle ABC, and intercepts chords A_1A_2, B_1B_2, C_1C_2 on BC, CA, AB respectively; from A_1 perpendiculars A_1b_1, A_1c_1 are dropped on CA, AB respectively, and from A_2, B_1, etc., similar perpendiculars. Show that the centres of the six circles, of which $A_1b_1c_1$ is one, lie on a circle concentric with the nine-point circle of ABC, and of radius half that of the original circle.

70. The symmedian AK meets BC in R, then $BR : RC = c^2 : b^2$.

71. If L, M, N be the projections on the sides of the symmedian point K, then K is the centroid of LMN.

72. The middle points of a side and of the corresponding altitude are collinear with the symmedian point.

73. The sides of the triangle LMN are perpendicular to the medians of ABC.

74. In the figure of Theorem 47 (Taylor circle) show that

(i) $Y'Z = Z'X = X'Y =$ perimeter of $\alpha\beta\gamma$;

(ii) X, X', Y, Y', Z, Z' are the feet of the perpendiculars from D, E, F on the sides of ABC;

(iii) YZ', ZX', XY', are third proportionals to BC and EF, CA and FD, AB and DE respectively.

CHAPTER III

HARMONIC RANGES AND PENCILS

Definitions

If A, B be two points in a straight line, and C, D two other points in the line, so taken that

$$\frac{AC}{BC} = -\frac{AD}{BD},$$

then the points A, C, B, D form a HARMONIC RANGE; and C and D are HARMONIC CONJUGATES with respect to A and B; AC, AB, AD are said to be in HARMONIC PROGRESSION; and AB is said to be the HARMONIC MEAN between AC and AD. [That these latter definitions agree with those given in algebra may be seen by writing the given proportion $\frac{AC}{BC} = -\frac{AD}{BD}$ in the form

$$\frac{AC}{AD} = \frac{CB}{BD} = \frac{AB - AC}{AD - AB}.]$$

In order that the above proportion may hold, one, and only one, of the points C, D must lie between A and B.

Theorem 1

If C *and* D *are harmonic conjugates with respect to* A *and* B, *then* A *and* B *are harmonic conjugates with respect to* C *and* D.

A O C B D

Since $\frac{AC}{BC} = -\frac{AD}{BD}$, $\therefore \frac{AC}{AD} = -\frac{BC}{BD}$, $\therefore \frac{CA}{DA} = -\frac{CB}{DB}$.

Theorem 2

If A, C, B, D *form a harmonic range, then*

$$\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}.$$

Since $$\frac{AC}{BC} = -\frac{AD}{BD};$$

$$\therefore \frac{AC}{AC - AB} = -\frac{AD}{AD - AB}, \quad \text{or} \quad \frac{AC - AB}{AC} = -\frac{AD - AB}{AD};$$

$$\therefore 2 = \frac{AB}{AC} + \frac{AB}{AD}; \quad \text{that is,} \quad \frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD}.$$

Theorem 3

If O *be the middle point of* AB, *then*

$$OA^2 = OB^2 = OC \cdot OD;$$

also $$DA \cdot DB = DC \cdot DO.$$

Since $$\frac{AC}{BC} = -\frac{AD}{BD}; \quad \therefore \frac{AC}{CB} = \frac{AD}{BD};$$

$$\therefore \frac{AC + CB}{AC - CB} = \frac{AD + BD}{AD - BD};$$

$$\therefore \frac{2OB}{2OC} = \frac{2OD}{2OB} \text{ (Introduction (4) [iv])};$$

or $$OC \,.\, OD = OB^2.$$

Again, $$\frac{DC - DA}{DC - DB} = -\frac{DA}{DB};$$

$$\therefore DC \,.\, DB - DA \,.\, DB = -DC \,.\, DA + DA \,.\, DB;$$
$$\therefore DC(DA + DB) = 2DA \,.\, DB,$$

or $$DC \,.\, DO = DA \,.\, DB.$$

Cor.—From the relation $OC \,.\, OD = OA^2$ it is obvious that if C coincides with O, the middle point of AB, then D, the harmonic conjugate of C with respect to A and B, is at an infinite distance.

Theorem 4

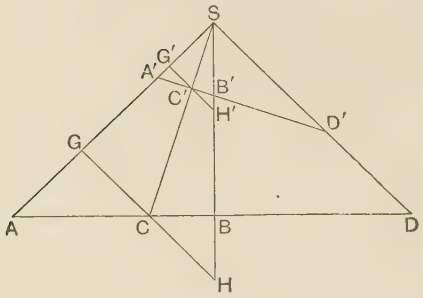
If S *be a point without the straight line in which is the harmonic range* ACBD, *and if through* C *a line be drawn parallel to* SD, *meeting* SA, SB *in* G, H *respectively, then* GC = CH.

Since $$\frac{AC}{BC} = -\frac{AD}{BD};$$

$$\therefore \frac{AC}{AD} = \frac{CB}{BD};$$

$$\therefore \frac{GC}{SD} = \frac{CH}{SD} \text{ (by similar triangles),}$$

or $$GC = CH.$$



Theorem 5

If in the figure of the last proposition any other transversal be drawn meeting SA, SB, SC, SD *in* A′, B′, C′, D′ *respectively, then* A′C′B′D′ *is a harmonic range.*

Through C′ draw G′C′H′ parallel to SD or GH.

Then, since $GC = CH$, $\therefore G'C' = C'H'$;

$$\therefore \frac{G'C'}{SD'} = \frac{C'H'}{SD'}; \quad \therefore \frac{A'C'}{A'D'} = \frac{C'B'}{B'D'};$$

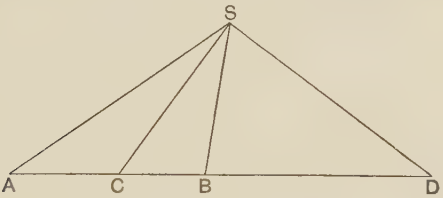
or $$\frac{A'C'}{B'C'} = -\frac{A'D'}{B'D'}.$$

DEFS.—A system of four straight lines SA, SC, SB, SD met by a transversal ACBD, so that ACBD is a harmonic range, is called a HARMONIC PENCIL; the point S is called the VERTEX of the pencil; and each of the four straight lines is called a RAY; SC, SD are said to be HARMONIC CONJUGATES of SA, SB.

Theorem 6

The legs of any angle form, with its internal and external bisectors, a harmonic pencil.

Let SC, SD be the internal and external bisectors of the angle ASB. Draw any transversal ACBD.



Then
$$\frac{AC}{CB} = \frac{AS}{SB} = \frac{AD}{BD};$$
$$\therefore \frac{AC}{BC} = -\frac{AD}{BD}.$$

Theorem 7

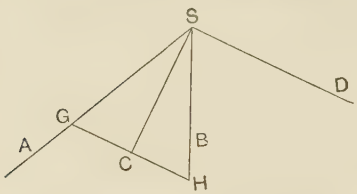
If in a harmonic pencil one ray bisect the angle between the two rays not conjugate to it, then it is perpendicular to its conjugate.

With the figure of last Theorem, if SC bisect the angle ASB, then
$$\frac{AS}{SB} = \frac{AC}{CB} = -\frac{AC}{BC} = \frac{AD}{BD},$$
and $\therefore$ SD bisects the exterior angle between SA, SB, and is consequently perpendicular to SC.

Theorem 8

If in a harmonic pencil one pair of conjugate rays be at right angles, then these are the internal and external bisectors of the angle between the other pair.

Let SC, SD be at right angles. Draw GCH parallel to SD. Then GC = CH (Theorem 4); the angles at C are right angles, and SC is common to the triangles GSC, HSC;

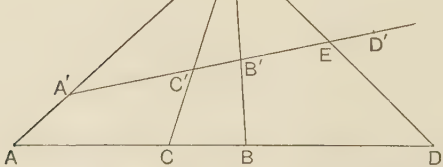


∴ SC bisects the angle GSH, and ∴ SD is external bisector.

Theorem 9

If two harmonic ranges are taken, one in each of two straight lines, and if three of the straight lines which connect corresponding points meet in a point, then the fourth connector passes through the same point.

Let ACBD, A′C′B′D′ be the two ranges, and let



AA′, CC′, BB′ meet in S; join SD, meeting A′C′B′ in E.

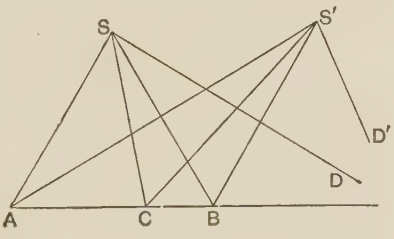
Then, since the pencil SA, SC, SB, SD is harmonic, $\therefore$ A′C′B′E is a harmonic range, but A′C′B′D′ is a harmonic range; $\therefore$ D′ and E coincide, or S, D′, D are collinear.

Cor.—If two corresponding points coincide in the intersection of the two lines, then the three connectors of the three other pairs of corresponding points are concurrent.

Theorem 10

If two harmonic pencils are such that the three points of intersection of three pairs of corresponding rays are in a straight line, then the point of intersection of the fourth pair of rays is in the same straight line.

Let S(ACBD), S′(ACBD′) be two harmonic pencils, ACB being collinear; then both SD and S′D′ must meet ACB in the point which is the harmonic conjugate of C with respect to A and B, which proves the theorem.



Cor.—If two of the rays, one in each pencil, are coincident, then the three points of intersection of the other three pairs of rays are collinear.

Examples on Chapter III

1. Three points A, B, C being in the same straight line, find two points equidistant from C which divide the segment AB harmonically.

2. If C, D are harmonic conjugates of A, B, then

$$AB \cdot CD + 2AD \cdot BC = 0,$$

and

$$AB \cdot DC + 2AC \cdot BD = 0.$$

3. Show how to find two points that are harmonic conjugates with respect to A, B, and also with respect to another pair of points C, D.

4. With the notation of Chapter II., show that AH, AI, AO, AI_2 form a harmonic pencil.

5. If A′, B′, C′ are the middle points of the sides BC, CA, AB of a triangle, then A′C′, A′B′ are harmonic conjugates with respect to A′A, A′C.

6. If a circle touch the sides BC, CA, AB of a triangle in X, Y, Z respectively, and YZ meet BC in X′, then X, X′ are harmonic conjugates with respect to B, C.

7. If through the intersection of the diagonals of a parallelogram straight lines are drawn parallel to the sides, they form with the diagonals a harmonic pencil.

8. If P, Q, R are collinear points in the sides BC, CA, AB of a triangle, and P′, Q′, R′ their harmonic conjugates with respect to those sides, then AP′, BQ′, CR′ are concurrent ; and conversely.

9. If AB is divided harmonically in C, D, and also in P, Q, and M, R, E are the middle points of AB, PQ, CD,

then

$$PC \cdot PD = 2PM \cdot RE,$$

and

$$QC \cdot QD = 2QM \cdot RE.$$

10. The tangent to the circumcircle at A and the symmedian AK are harmonic conjugates with respect to the sides AB, AC of the triangle ABC.

11. When four points on a circle determine a harmonic pencil of rays at any fifth point on the circle, they do so at every point of the circle.

12. When four tangents to a circle determine a harmonic range on any fifth tangent, they do so on every tangent to the circle.

13. When four points on a circle form a harmonic system, the tangent at any one forms a harmonic pencil with its joins with the other three.

14. When four tangents to a circle form a harmonic system, the point of contact of any one forms a harmonic range with its intersections with the other three.

15. A circle is described touching a straight line AB at C, and having its diameter a harmonic mean between the segments AC, CB. Prove that it touches the circle on AB as diameter.

CHAPTER IV

PROPERTIES OF A COMPLETE QUADRILATERAL AND COMPLETE QUADRANGLE

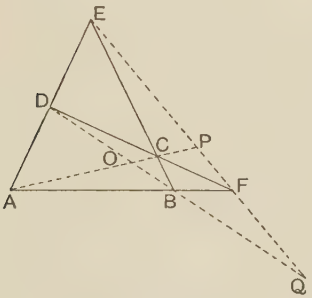
Definitions

A SYSTEM of four straight lines (no three of which are concurrent) in which the six points of intersection are taken into account is called a COMPLETE QUADRILATERAL. The four lines are called the SIDES of the quadrilateral, and the six points the VERTICES. Two vertices which do not lie on the same "side" are called OPPOSITE VERTICES. The three straight lines joining "opposite vertices" are called DIAGONAL LINES, and

A system of four points (no three of which are collinear) in which the six straight lines of connection are taken into account is called a COMPLETE QUADRANGLE. The four points are called VERTICES of the quadrangle, and the six straight lines the SIDES. Two sides which do not pass through the same "vertex" are called OPPOSITE SIDES. The three points of intersection of "opposite sides" are called DIAGONAL POINTS,

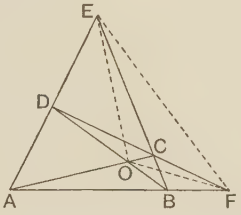
the triangle formed by the "diagonal lines" is called the DIAGONAL TRIANGLE of the complete quadrilateral.

Thus AB, BC, CD, DA are the four straight lines; A, B, C, D, E, F are the six vertices; EF, AC, BD are the three diagonals; and OPQ is the diagonal triangle.



and the triangle formed by joining the "diagonal points" is called the DIAGONAL TRIANGLE of the complete quadrangle.

Thus A, B, C, D are the four points; AB, BC, CD, DA, AC, BD are the six sides; E, O, F are the three diagonal points; and EOF is the diagonal triangle.



The above is Cremona's notation. It will be observed that

a complete quadrilateral with its diagonal triangle consists of seven *straight lines* and nine *points of intersection.*

a complete quadrangle with its diagonal triangle consists of seven *points* and nine *connecting lines.*

Theorems 1 *and* 2

In a complete quadrilateral *each pair of opposite* vertices *forms a harmonic* range *with two of the angular* points *of the diagonal triangle.*

In a complete quadrangle *each pair of opposite* sides *forms a harmonic* pencil *with two of the* sides *of the diagonal triangle.*

Using the above figures, we have a triangle EAF, in which FB, AP, FD are concurrent, and DBQ is a transversal;

$$\therefore \text{AB} \cdot \text{FP} \cdot \text{ED} = \text{BF} \cdot \text{PE} \cdot \text{DA},$$

and

$$\text{AB} \cdot \text{FQ} \cdot \text{ED} = -\text{BF} \cdot \text{QE} \cdot \text{DA};$$

$$\therefore \text{PF} : \text{QF} = -\text{PE} : \text{QE};$$

$\therefore$ E, P, F, Q form a harmonic range,

and $\therefore$ OE, OC, OF, OB form a harmonic pencil.

In like manner, by considering the triangles EAC, EDB, we find that AOCP and DOBQ are harmonic ranges, and ED, EO, EC, EF, and FB, FO, FC, FE are harmonic pencils.

Cor.—Every range of four points, and every pencil of four rays in the foregoing figures, are harmonic.

We may use these theorems to solve by means of the ruler only the following problems:—

Problems

Given three points of a harmonic range to find the fourth.

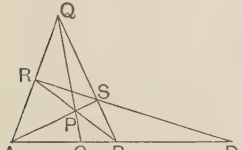
Given three rays of a harmonic pencil to construct the fourth.

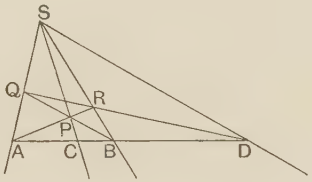
Let A, C, B be the given points.

Draw any two straight lines through A, and let a line through C cut them in P and Q. Join BP, BQ, and let them cut AQ, AP respectively in R and S, and let RS meet AB in D, then D will be the required point.

Let SA, SC, SB be the given rays.

Take any two points A and Q in SA and join them to any point P in SC, and let AP and QP cut SB in R and B respectively, and let QR meet AB in D, then SD will be the required ray.





Principle of Duality

The definitions relating to a complete quadrilateral and complete quadrangle are such that each can be obtained from the other by the interchange of the terms "point" and "line," "vertex" and "side." The reason of this is that in the first case the figure is regarded as consisting of a system of straight lines which intersect in points, whilst in the latter we regard it as a system of points which are connected by straight lines, just in the same way as a curve may be considered either as the envelope of a series of straight lines or as a system of points.

Introducing the idea of harmonic section we obtain Theorem 1 or Theorem 2 according as we take lines or points as elements in the description of the figures, and each Theorem can be obtained from the other by interchanging the terms "point" and "line," "range" and "pencil."

It will be seen that every descriptive theorem or theorem of position is such that from it can be derived another corresponding or correlative theorem by means of the above and similar changes. Thus corresponding to "concurrent lines" we have "collinear points," to "line joining two points" we have "point of intersection of two lines," to "points forming a harmonic range" we have "lines forming a harmonic pencil," etc. This correlation of "point" and "line" is termed the "principle of duality." Each of two descriptive theorems so correlated is said to be the dual of the other, and it will be found that if any descriptive property is demonstrated, its dual also holds.

Theorem 3

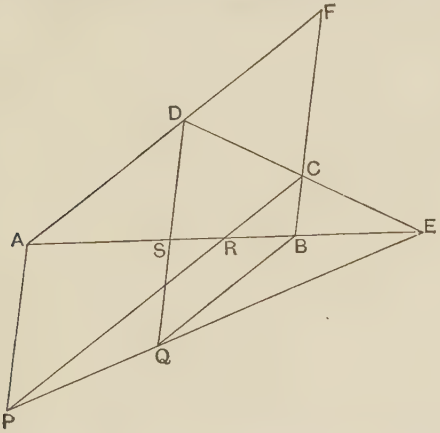
The midpoints of the segments of the three diagonal lines of a complete quadrilateral intercepted between opposite vertices are collinear.

Let ABECFD be the complete quadrilateral, then the midpoints of AC, BD, EF are collinear.

Complete the parallelograms AFCP, BFDQ, and join PQ, QE. Let AE meet CP, DQ in R, S.

Then
$$\frac{EB}{BS}=\frac{EC}{CD}=\frac{ER}{RA};$$
$$\therefore \frac{EB}{ER}=\frac{BS}{RA}=\frac{BQ}{RP};$$
$$\therefore \text{ P, Q, E are collinear;}$$

$\therefore$ the midpoints of FP, FQ, FE lie on a line parallel to PQE; and these are the midpoints of AC, BD, and



EF, since the diagonals of a parallelogram bisect each other.

Examples on Chapter IV

1. In the figure of Theorem 1, show that PD, QA, OF are concurrent.

2. In the same figure, show that the points of intersection of PD, OF and PB, OE are collinear with A and Q.

3. In the same figure, if X, Y, Z are the middle points of DB, AC, EF respectively, prove that $XO : XQ = DO^2 : DQ^2$, and corresponding relations for Y and Z; hence show that X, Y, Z are collinear.

4. In the same figure, prove that the circumcircles of the triangles EAB, ECD, FBC, FDA meet in a point R, and that $RB \cdot RD = RE \cdot RF$.

5. Show that in the last example all the triangles have the same Simson line with respect to the point R, and that the orthocentres of the triangles are collinear.

CHAPTER V

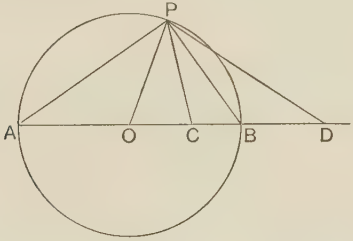
PROPERTIES OF A CIRCLE

Theorem 1

If C, D *be harmonic conjugates with respect to* A, B, *the extremities of a diameter of a circle, and if* P *be any point on the circumference, then, regarding only the magnitudes of the lines,*

$$\frac{PC}{PD} = \frac{AC}{AD} = \frac{BC}{BD}.$$

Let O be the centre of the circle, then



$OC \cdot OD = OA^2 = OP^2$ (Chap. III. Theorem 3);

$$\therefore \frac{OC}{OP} = \frac{OP}{OD};$$

$\therefore$ the triangles OCP and OPD are similar, and

$$\therefore \frac{PC}{PD} = \frac{OC}{OP} = \frac{OP}{OD}$$
$$= \frac{OC + OP}{OP + OD} = \frac{OP - OC}{OD - OP}$$
$$= \frac{AC}{AD} = \frac{BC}{BD}.$$

This theorem also follows at once from Chap. III. Theorem 8.

Cor.—Since the segments of the base BC, BD are as the sides PC, PD of the triangle PCD, $\therefore$ PB bisects the angle CPD, and similarly PA bisects the exterior angle between CP and PD.

Theorem 2

If C, D *be two fixed points, and* P *be a point such that the ratio* PC : PD *is constant, then the locus of* P *is a circle, except when the constant is unity, and then the locus becomes a straight line.*

Divide CD (see fig. to last proposition) internally and externally at B and A in the given ratio ; then, since

$$\frac{PC}{PD} = \frac{BC}{BD} = \frac{AC}{AD};$$

$\therefore$ PB and PA bisect the interior and exterior angles between PC and PD, and consequently are at right angles, and

$\therefore$ P must lie on the circle described on AB as diameter.

It is obvious, if the ratio PC : PD is unity, that P is on the line which bisects CD at right angles.

INVERSION

DEF.—If C, D be harmonic conjugates with respect to A, B, the extremities of any diameter of a given circle, then C, D are termed INVERSE POINTS with respect to the circle.

The term *inverse* comes from the fact that if O be the centre of the circle, $OC \cdot OD = OA^2$, and consequently OC, OD vary inversely.

It is clear that the position of the centre of the circle and the square of its radius are necessary and sufficient to determine the inverse of any point; the former is called the CENTRE OF INVERSION, and the latter the CONSTANT.

If we obtain the inverse points of every point on a line or configuration, all the inverse points will form another line or configuration, which is called the inverse of the former.

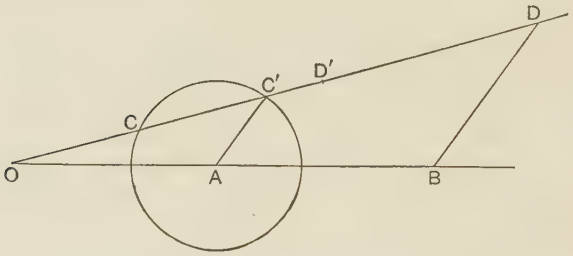
We shall first establish some important properties of inverse figures, and then give some examples of the use of the method of inversion.

Theorem 3

If C, D *be inverse points with respect to a circle whose centre is* O, *and one of the two points* C *describe a circle, the inverse point* D *will describe another circle, except when*

the circle which C *describes passes through* O, *in which case* D *will describe a straight line.*

Let A be the centre of the circle on which C lies,

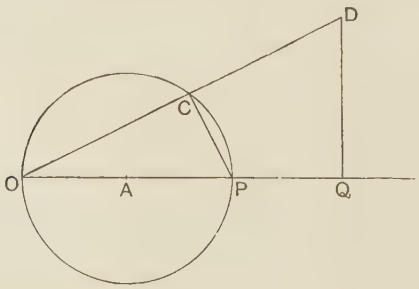


and let OCD meet this circle again in C′. Draw DB parallel to AC′, meeting OA in B.

Then OC . OD is constant, and so is OC . OC′;

$\therefore \dfrac{OD}{OC'}$ is a constant ratio, and is equal to $\dfrac{OB}{OA}$ and $\dfrac{BD}{AC'}$;

$\therefore$ B is a fixed point and BD a fixed length; that is, D lies on a circle whose centre is B.



If the circle on which C lies passes through O, draw its diameter OP, and let Q be the point inverse to P, so that

$$OP \cdot OQ = OC \cdot OD,$$

or

$$\frac{OP}{OC} = \frac{OD}{OQ};$$

$\therefore$ the triangles OCP, OQD are similar, and $\therefore$ Q is a right angle; and D is on the straight line through Q at right angles to OQ.

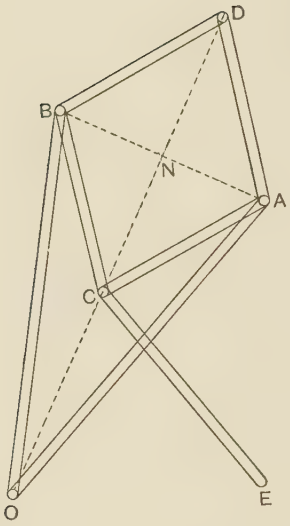
NOTE.—It will be observed that the circle with regard to which C and D are inverse points is not drawn in the figures of this theorem; and generally in questions of inversion it is sufficient to realise that the rectangle $OC \cdot OD = R^2$, a constant, and C and D are said to be inverse points with respect to O. Also it may be observed that C and D may be on opposite sides of O, in which case $OC \cdot OD = -R^2$, and the circle is said to be imaginary.

It is from the principle of inversion, in the latter portion of Theorem 3, that Peaucellier invented the system of linkages by which rectilinear motion may be derived from circular motion.

The following is one of the simplest methods: —

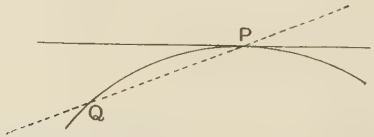
A rhombus ACBD, freely jointed at its angular points, is attached at A and B to two equal rods AO, BO movable about O, which is fixed. Then, however the rods be moved, O, C, D will always be collinear. Also, since the diagonals AB, CD bisect at right angles,

$$\begin{aligned} OC \cdot OD &= ON^2 - CN^2 \\ &= OB^2 - CB^2 \\ &= \text{constant}; \end{aligned}$$



∴ C and D are inverse points with respect to O. Consequently, if C be made to describe a circle passing through O by means of a rod CE, E being fixed so that EO = EC, then D will move along a straight line at right angles to OE.

DEF.—If P and Q be two points near one another on a curve, the position taken up by the straight line PQ when Q moves up indefinitely near to P is called the TANGENT TO THE CURVE at the point P.



DEF.—The angle made by a straight line and a curve is the angle between the straight line and the tangent to the curve at the point of intersection.

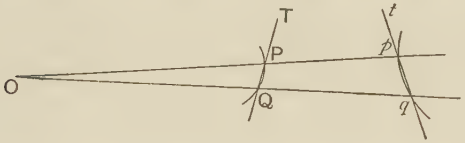
DEF.—The angle between two curves that intersect

is the angle between the tangents at the point of intersection.

Theorem 4

Every straight line through the centre of inversion cuts a curve and its inverse at supplementary angles.

Let P and Q be two points near one another on a



curve, and p and q the corresponding points on the inverse curve. Then

$$\mathrm{OP} \,.\, \mathrm{O}p = \mathrm{OQ} \,.\, \mathrm{O}q = \text{constant};$$

$\therefore$ P, p, q, Q are concyclic; and the angles QPp or OPT, and Qqp are supplementary.

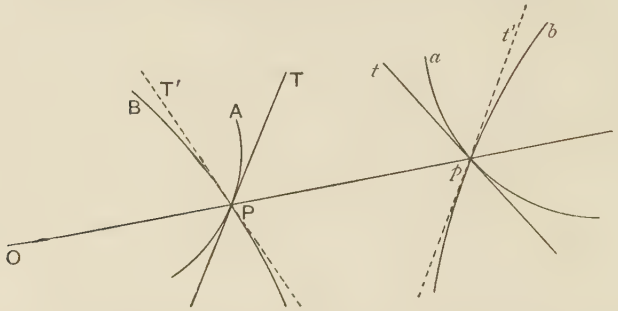
Now let Q move close up to P, then q will move close up to p, and the straight lines QPT, qpt become the tangents to the curves at P and p, and the supplementary angles become OPT and Opt, which are the angles made by OPp with the tangents; hence the required result.

Theorem 5

Two loci cut at the same angle as their inverses cut.

Let PT, PT′ be tangents to the curves A, B at their

point of intersection P. Then the inverse point p, being on both the inverse curves a and b, must be their



point of intersection; let pt, pt' be the tangents to the inverse curves a and b.

By Theorem 4

$$\text{angle OPT} = 180° - Opt,$$

and $$\text{OPT}' = 180° - Opt';$$

$$\therefore \text{TPT}' = tpt';$$

$\therefore$ the curves A and B intersect at the same angle as their inverses a and b.

Cor.—If two curves touch each other at a point, their inverses also touch at the inverse of that point.

Theorem A

If ACBD *be a harmonic range, then the inverse points* A', C', B', D' *with regard to any point* O *on the range also form a harmonic range.*

Since $\frac{AC}{BC} = -\frac{AD}{BD}$, $\therefore \frac{OC - OA}{OC - OB} = -\frac{OD - OA}{OD - OB}$;

$$\therefore \frac{\frac{1}{OC'} - \frac{1}{OA'}}{\frac{1}{OC'} - \frac{1}{OB'}} = -\frac{\frac{1}{OD'} - \frac{1}{OA'}}{\frac{1}{OD'} - \frac{1}{OB'}};$$

$$\therefore \frac{OA' - OC'}{OB' - OC'} = -\frac{OA' - OD'}{OB' - OD'};$$

that is, $$\frac{C'A'}{C'B'} = -\frac{D'A'}{D'B'}, \quad \text{or} \quad \frac{A'C'}{B'C'} = -\frac{A'D'}{B'D'}.$$

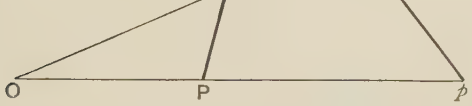
Theorem B

If P, p *and* Q, q *are pairs of inverse points,* O *the centre of inversion, and* R^2 *the constant, then*

$$pq : PQ = R^2 : OP \cdot OQ.$$

We have $OP \cdot Op = R^2 = OQ \cdot Oq$;

$\therefore$ the triangles POQ, qOp have a common angle O, and $OP : OQ = Oq : Op$;



$\therefore$ the triangles are similar;

$\therefore pq : PQ = Op : OQ$

$= OP \cdot Op : OP \cdot OQ$

$= R^2 : OP \cdot OQ.$

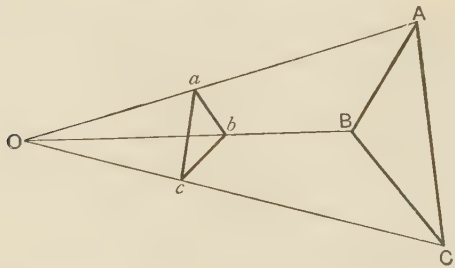
This result is of importance, as it gives the ratio of the linear dimensions of a figure and its inverse.

Examples of Inversion

We now proceed to give two examples of the method of inversion as applied to plane geometry ; the method is also frequently made use of in certain problems on attractions, the mathematical theory of electricity, etc.

(1) *To prove from* Euclid I. 20 *that the rectangle under the diagonals of any quadrilateral is less than the sum of the rectangles under its opposite sides, excepting the case of a cyclic quadrilateral.*

Let ABC be a triangle, and a, b, c the inverse points



of A, B, C with respect to any centre O, and constant R^2.

Then
$$ab = \frac{R^2}{OA \cdot OB} \cdot AB$$
$$bc = \frac{R^2}{OB \cdot OC} \cdot BC$$

$$ca = \frac{R^2}{OC \cdot OA} \cdot CA,$$

and $$OA \cdot Oa = OB \cdot Ob = OC \cdot Oc = R^2;$$

$$\therefore ab \cdot Oc + bc \cdot Oa = \frac{R^4 \cdot AB}{OA \cdot OB \cdot OC} + \frac{R^4 \cdot BC}{OA \cdot OB \cdot OC}$$

$$= \frac{R^4}{OA \cdot OB \cdot OC} \left\{ AB + BC \right\};$$

$$\therefore ab \cdot Oc + bc \cdot Oa > \frac{R^4 \cdot AC}{OA \cdot OB \cdot OC}, \quad \text{or} \quad \frac{R^2}{OB} \cdot \frac{R^2 \cdot AC}{OA \cdot OC};$$

$$\therefore ab \cdot Oc + bc \cdot Oa > Ob \cdot ac.$$

Which proves the proposition for the quadrilateral $Oabc$.

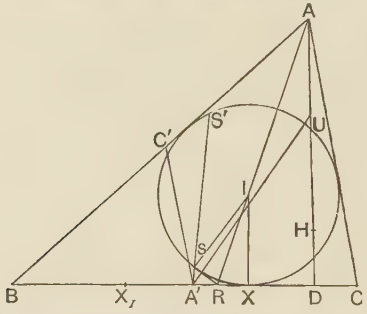
If we take the point B on the straight line AC, then $AB + BC = AC$, also the straight line ABC inverts into a circle $Oabc$, and the inequality becomes an equality, so that *in a cyclic quadrilateral the rectangle under the diagonals is equal to the sum of the rectangles under the opposite sides.* This result is generally known as Ptolemy's Theorem.

(2) *To prove, by the method of inversion, that the nine-point circle of a triangle touches the inscribed and escribed circles* (*Feuerbach's Theorem*).

Let AR be the bisector of the angle A, then, with the notation of Chap II., it can be shown by Chap. III. Theorem 6, that $AIRI_1$ is a harmonic range, $\therefore$ by parallels, $DXRX_1$ is a harmonic range; and A′ is the midpoint of XX_1,

$$\therefore A'R \cdot A'D = A'X^2 = A'X_1^2.$$

If we invert with respect to centre A′, and constant $A'X^2$, the incircle and excircle touching BC at X and X_1 invert into themselves, and the nine-point circle inverts



into a straight line through R at right angles to its diameter A′U, and therefore making with BC an angle

$$= A'UD = A'C'D \text{ in same segment}$$
$$= BC'D - BC'A' = 2BAD - BAC = C - B.$$

Now RS, the tangent from R to the incircle, which also touches the excircle, makes with BC an angle

$$= XIS = 2XIR = 2RAD = C - B,$$

therefore RS is the inverse of the nine-point circle, and it touches the inverses of the incircle and the excircle touching BC, therefore the nine-point circle touches the incircle and excircles. The point of contact with the incircle is S′, the inverse of S, *i.e.* the point where A′S again cuts the incircle. The point of contact with the excircles could be found in like manner.

The above proof first appeared in the *Quarterly*

Journal of Mathematics, vol. xiii., and is due to Mr. J. P. Taylor.

POLES AND POLARS

DEF.—If C, D be harmonic conjugates with respect to A, B, the extremities of a diameter of a circle, and if through D a line DE be drawn parallel to the tangent at A or B, then DE is called the POLAR of C, and C the POLE of DE with respect to the circle.

It is clear that if the pole is within the circle its polar is without, and *vice versâ*; also as C approaches O, the centre of the circle, its polar recedes from O and *vice versâ*; the polar of O is at infinity, and the polar of a point on the circumference is the tangent at the point.

HARMONIC PROPERTY OF POLE AND POLAR

Theorem 6

Any straight line through the pole is cut harmonically by the pole, its polar, and the circumference.

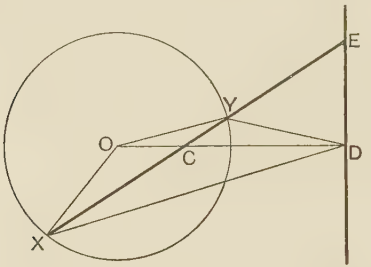
Let C be the pole and DE its polar, and let a straight line through C cut the circle in X, Y, and the polar in E. Suppose C within the circle.

Then $OC \cdot OD = OY^2 = OX^2$;

$\therefore$ the triangles COY, YOD having a common angle O, and $OC : OY = OY : OD$ must be similar.

In like manner the triangles COX, XOD are similar;

$\therefore$ angle ODY = OYC by similar triangles
= OXC, $\because$ OX = OY
= ODX by similar triangles;
$\therefore$ DC bisects the angle XDY;



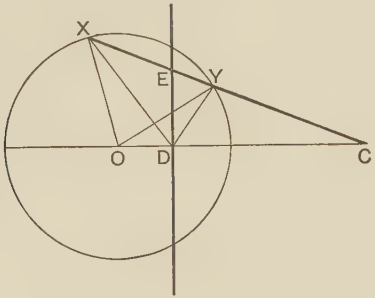
$\therefore$ DE, which is at right angles to DC, bisects the exterior angle between DX and DY;
$\therefore$ DX, DC, DY, DE form a harmonic pencil;
$\therefore$ X, C, Y, E form a harmonic range.

When C is without the circle we have pairs of similar triangles COY, YOD, and COX, XOD as before;

$\therefore$ angle ODX = OXC by similar triangles
= OYX, $\because$ OX = OY
= suppt. OYC
= suppt. ODY by similar triangles
= CDY;
$\therefore$ angle XDE = YDE,
and the result follows as before.

Cor.—If through any point C in the plane of a circle a straight line is drawn, cutting the circle in X and Y,

then the locus of the harmonic conjugate of C with respect to X and Y is the polar of C with respect to the circle.

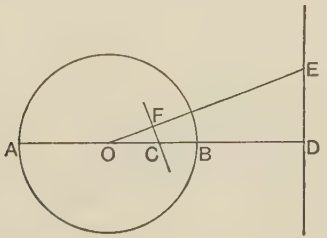


RECIPROCAL PROPERTY OF POLE AND POLAR

Theorem 7

If any point E *is taken on the polar* DE *of a point* C, *then the polar of* E *passes through* C.

Join OE and draw CF perpendicular to it, then C, D, E, F are concyclic;



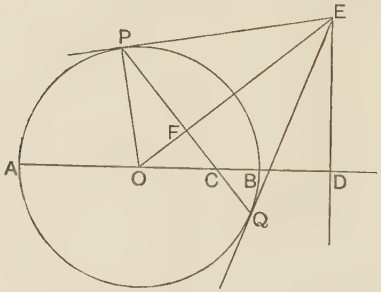
$\therefore$ OF . OE = OC . OD = OA^2;
$\therefore$ CF is the polar of E.

Cor.—If E′ be any other point on DE, then the polar of E′ passes through C. Hence C, the point of intersection of the polars of two points E, E′, is the pole of the line EE′ which joins them.

Theorem 8

The polar of any point with regard to a circle is the locus of the intersections of tangents at the extremities of chords through the point.

Let PCQ be any chord through C, and let the tan-



gents at P, Q intersect in E, and ED be perpendicular to OC. OE bisects PQ at right angles and

$$OP^2 = OF \cdot OE = OC \cdot OD$$

(since C, D, E, F are concyclic), and $\therefore$ DE is the polar of C.

COR.—PQ is obviously the polar of E; hence we see that the polar of a point external to the circle is the chord of contact of tangents drawn from that point.

DEF.—Any two points so related that each lies on the polar of the other with regard to a circle, are termed CONJUGATE POINTS with regard to the circle; and any two lines so related that each passes through the pole of the other with regard to a circle, are termed CONJUGATE LINES with regard to the circle.

Theorem C

Any pair of conjugate lines drawn through the intersection of two tangents to a circle form with the two tangents a harmonic pencil.

By Theorem 6 the four lines intercept on the chord of contact a harmonic range, and the proposition is thus only another way of stating the harmonic property of pole and polar.

DEF.—A triangle of which each side is the polar of the opposite vertex with regard to a circle is said to be SELF-CONJUGATE with respect to the circle.

Theorem 9

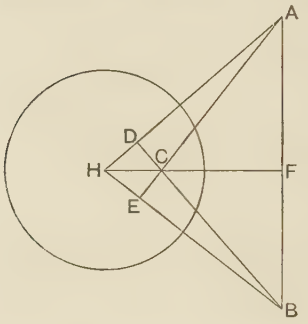
The centre of the circle to which a given triangle is self-conjugate is at the orthocentre of the triangle, and its radius is a mean proportional between HA *and* HD (*see notation of Chap. II.*)

Since the centre of the circle lies in the line through

the pole perpendicular to the polar, it is obvious that the centre of the circle to which the triangle ABC is self-conjugate must be H, and the square on its radius is equal to

$$HA \cdot HD = HB \cdot HE = HC \cdot HF.$$

NOTE.—Since the polar of a point within the circle lies entirely without it, it is obvious that one of the angular points of the self-conjugate triangle must be within the circle and the other two without it. Also, in order that there may be such a circle, the orthocentre must be external to the triangle—that is, the triangle must be obtuse-angled. In the case of an acute-angled triangle the circle is said to be imaginary (see note to Theorem 3).



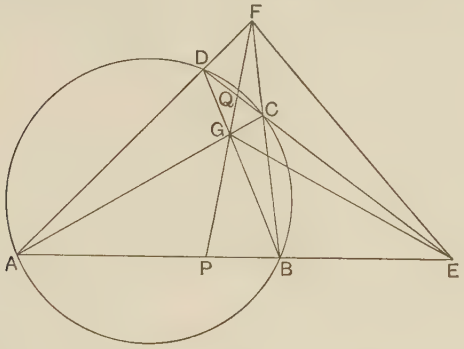
POLE AND POLAR PROPERTIES OF A QUADRANGLE

Theorem 10

If a complete quadrangle be inscribed in a circle, its diagonal triangle is self-conjugate with respect to the circle.

Let EFG be the diagonal triangle of the quadrangle ABCD, and let FG meet AB, CD in P and Q; then F (APBE) is a harmonic pencil (Chap. IV. Theorem 2),

and ∴ P is the harmonic conjugate of E with respect



to A and B, and ∴ the polar of E passes through P (Theorem 6).

Similarly it passes through Q; ∴ FG is the polar of E.

Similarly GE is the polar of F, and ∴ EFG is self conjugate with respect to the circle.

NOTE.—From this theorem we can by a ruler only draw the polar of any point E, thus:—Draw any two chords of the circle through E, EBA, ECD; then the join of the intersection F of AD and BC with G, the intersection of AC and BD, is the polar required.

Secondly, to find the pole of any straight line EF:—Take any two points E, F in the line, draw a chord EBA, and join FA, FB meeting the circle in D and C, the intersection G of AC and BD is the required pole.

Thirdly, to draw a tangent to the circle from E:—

Find the polar of E as above, then the joins of E with the points where this polar meets the circle are the required tangents (Theorem 8, Cor.)

Theorem 11

The diagonal triangle of a complete quadrangle inscribed in a circle is also the diagonal triangle of the complete quadrilateral whose sides touch the circle at the vertices of the quadrangle.

FG, the polar of E (see last fig.), passes through the pole of EBA (Theorem 7)—that is, through the intersection of the tangents at A and B.

Similarly FG passes through the intersection of the tangents at C and D;

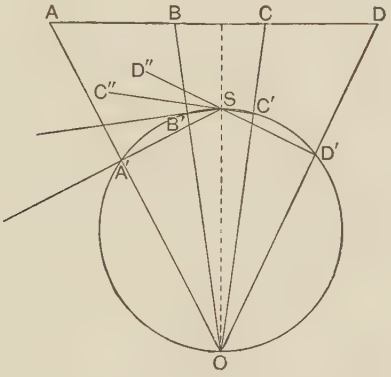
$\therefore$ FG is one of the sides of the diagonal triangle of the complete quadrilateral, whose sides touch the circle at A, B, C, D; and similarly for GE and EF.

Theorem 12

If four points form a harmonic range, their polars form a harmonic pencil, and conversely the poles of the rays of a harmonic pencil form a harmonic range.

Let ABCD be a harmonic range, and O the centre of the circle with regard to which the polars are to be taken. The inverse points A′, B′, C′, D′ lie on a circle

passing through O, and whose diameter OS is perpendicular to the range ABCD. Join SA′, SB′, SC′, SD′; then these lines, being at right angles to OA′, OB′, OC′, OD′, are the polars of A, B, C, D, and the angles



between them are equal to the angles between OA, OB, OC, OD, the rays of a harmonic pencil, therefore they form a harmonic pencil.

Conversely, if SA′, SB′, SC″, SD″ be the rays of the given harmonic pencil, the poles of these lines lie on a straight line ABCD, the inverse of the circle on SO as diameter; and the angles between the rays of the pencils whose vertices are S and O being equal, since one is a harmonic pencil, the other must be harmonic also, therefore A, B, C, D form a harmonic range.

Examples on Chapter V

1. If P, P′ and Q, Q′ are pairs of inverse points in the same diameter, and X any point on the circle of inversion, then the angles PXQ, P′XQ′ are equal.

2. If P, P′ be inverse points with respect to a circle whose centre is C, and X is any point on the circle, then

$$PX^2 : P'X^2 = PC : P'C.$$

3. Show how to invert a circle into a given circle.

4. The inverse of the centre of inversion with respect to any circle inverts into the centre of the circle inverse to that circle ; and conversely, the centre of any circle inverts into the inverse of the centre of inversion with respect to the inverse circle.

5. If A, B, C, D be four collinear points, and A′, B′, C′, D′ their inverses, then

$$\frac{AC \cdot BD}{AB \cdot CD} = \frac{A'C' \cdot B'D'}{A'B' \cdot C'D'}$$

6. Three circles have a common point of intersection ; show by inversion that the circles drawn through this point and the other point of intersection of any two of the circles, at right angles to the third circle, all meet in a second point.

7. Invert the theorem that if a harmonic pencil issues from a point A, the range made by the four rays on any straight line passing through a fixed point B is harmonic ; taking B for centre of inversion.

8. Invert the theorem that the angle in a given segment of a circle is constant.

9. Invert the theorem that the opposite angles of a cyclic

quadrilateral are supplementary, with respect to (i) a corner of the quadrilateral, (ii) a point on the circle, (iii) a point outside the circle.

10. Invert Menelaus' Theorem with respect to a vertex of the triangle.

11. Prove by inversion that the circles having for diameters three chords OA, OB, OC of a circle intersect again by pairs in three collinear points.

12. Prove by inversion that if three circles meet by pairs in A, A′; B, B′; C, C′, and O be any point in their plane, the circles OAA′, OBB′, OCC′ all meet in another point.

13. If a point lie on a fixed circle, its polar with regard to a concentric circle always touches another concentric circle.

14. From a given point P in a fixed radius of a given circle PA, PB are drawn to the circumference, on the same side of, and equally inclined to the fixed radius; prove that AB passes through a fixed point.

15. A tangent being drawn to a circle, prove that the point of contact, and the points where it crosses any two other tangents and their chord of contact, form a harmonic range.

16. The circle whose diameter is the distance between the centres of inversion and of any circle, inverts into the polar of the centre of inversion with respect to the inverse circle.

17. The distances of two points from the centre of a circle have the same ratio as their distances each from the polar of the other with respect to the circle.

18. The lines joining the extremities of any chord of a circle to the middle point of a chord through its pole are equally inclined to the latter chord.

19. The circle with respect to which a triangle is self-conjugate cuts the sides of the triangle harmonically.

20. If the vertices of the triangle ABC be the poles of the sides of a triangle PQR, then the vertices of PQR are the poles of the sides of ABC.

21. If two points are conjugate with respect to a circle, the square of the distance between them is equal to the sum of the squares of the tangents from them to the circle.

22. The inverse of the circumcircle, with respect to the circle with regard to which the triangle is self-conjugate, is the nine-point circle.

23. If A and B be points on concentric circles, then the perpendicular from B on the polar of B with respect to the circle through A, is to the perpendicular from A on the polar of A with respect to the circle through B, in the ratio of the radii.

24. Through a fixed point O any two chords POP′, QOQ′ are drawn cutting a fixed circle, show that the locus of the second point of intersection of the circles POQ, P′OQ′ is a circle. [Invert with respect to O.]

25. If T be a point outside or inside a circle, and TU perpendicular to the polar of T, and A, B any two points on the circle, then

$$AT : BT = AU : BU.$$

CHAPTER VI

PROPERTIES OF TWO OR MORE CIRCLES

ORTHOGONAL CIRCLES

Definition

Two circles which cut each other so that the tangents at their points of section are at right angles, are said to be ORTHOGONAL.

Theorem 1

If two circles are orthogonal, the tangent to either at a point of section passes through the centre of the other.

This is obvious, since a diameter is perpendicular to the tangent at its extremity.

This theorem might also be stated thus: In a pair of orthogonal circles the radii drawn to a point of section are at right angles.

Cor.—If two circles are orthogonal, the square on the distance between their centres is equal to the sum of the squares on the radii, and conversely.

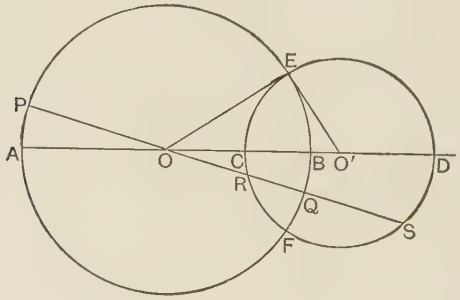
Theorem 2

If two points C, D *are harmonic conjugates with respect to two others* A, B, *the circles described on* AB *and* CD *as diameters are orthogonal.*

Let O, O′ be the centres of the circles which cut at E, F; then

$$OC \cdot OD = OA^2 = OE^2,$$

and ∴ OE touches the circle on CD, and ∴ the circles are orthogonal.



Theorem 3

If two circles are orthogonal, every diameter of either which meets the other is cut harmonically.

Let the diameter POQ meet the other circle in R, S (see last fig.)

Then, since OE touches this circle,

$$OR \cdot OS = OE^2 = OP^2;$$

∴ PRQS is a harmonic range.

COR.—If P, Q and R, S are pairs of harmonic conjugates, then any circle through the points R, S is cut orthogonally by the circle on PQ as diameter.

Theorem 4

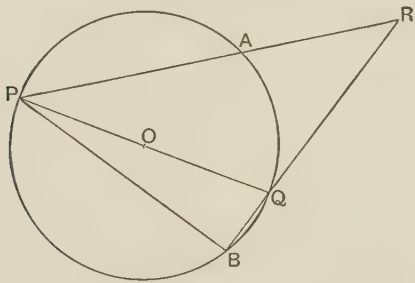
Circles are orthogonal if the angles in the segments on opposite sides of the chord of intersection are complementary.

If the angles EAF, EDF are complementary, then EOF, EO′F must be supplementary; consequently OEO′ and OFO′ are supplementary, and being equal, each is a right angle; and ∴ the circles are orthogonal.

Theorem 5

The locus of the points of intersection of the straight lines joining two fixed points on a circle to the extremities of a variable diameter is the circle through the fixed points orthogonal to the given circle.

Let A, B be the two fixed points, POQ any diameter,



and let PA, QB intersect in R. Join PB. Then, since

PQ is a diameter, PBQ is a right angle, and ∴ BPR, BRP are complementary; ∴ the locus of R is a circle on AB containing an angle complementary to APB, and ∴ cutting the given circle orthogonally (Theorem 4).

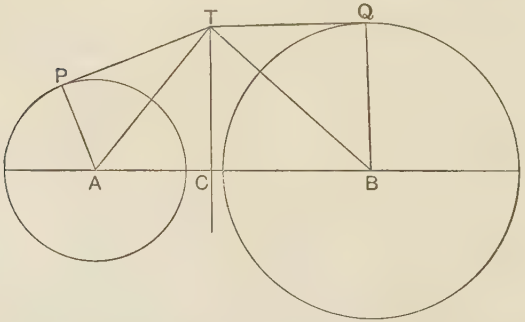
It should be noted under what circumstances R will describe the portion of the circle which is within the given circle.

RADICAL AXIS

Theorem 6

The locus of a point from which tangents drawn to two given circles are equal is a straight line at right angles to the line joining the centres of the circles, and dividing this line so that the difference of the squares on the segments is equal to the difference of the squares on the radii of the circles.

Let A, B be the centres of the two circles, and let



the tangents TP, TQ be equal. Join AP, AT, BQ, BT, and draw TC perpendicular to AB.

Then $BC^2 - AC^2 = BT^2 - AT^2$
$= BQ^2 + TQ^2 - (AP^2 + TP^2)$
$= BQ^2 - AP^2$.

Since $BC^2 - AC^2 = AB(BC - AC)$, it is obvious that C is a fixed point; hence the required result.

COR.—If the two circles intersect, the points of intersection are obviously on this locus, which is therefore the common chord of the two circles.

DEF.—The straight line which is the locus of the point from which tangents drawn to two given circles are equal is called the RADICAL AXIS of the two circles.

Theorem 7

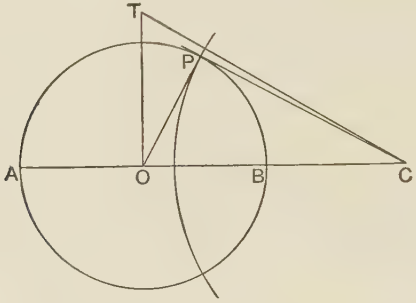
If circles be drawn having their centres in a fixed diameter (produced) of a given circle, and orthogonal to it, the radical axis of any pair of the circles is the straight line through the centre of the given circle at right angles to the fixed diameter (see fig., next page).

Let C be the centre of one of the circles, in the fixed diameter AOB of the given circle, P a point of intersection of the two circles, T any point in the line through O perpendicular to AB. Then the square on the tangent from T to the circle whose centre is C

$= CT^2 - CP^2 = OT^2 + OC^2 - CP^2 = OT^2 + OP^2 = AT^2$,

which is constant for all positions of C, and $\therefore$ OT is the radical axis of every pair of such circles.

COR.—Since OT lies outside all the circles drawn orthogonal to the given circle, it is obvious that none of these circles intersect.



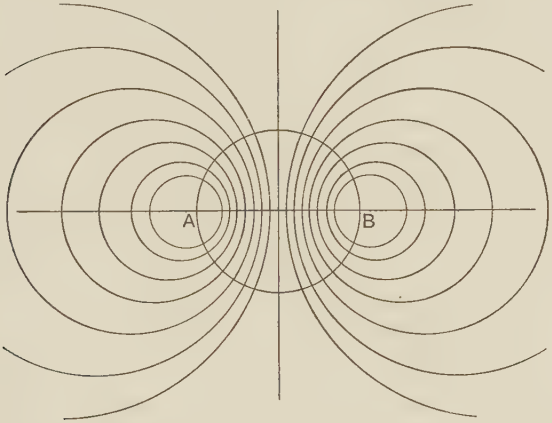
DEF.—If a system of circles be such that every pair has the same radical axis, then the circles are said to be CO-AXAL.

In Theorem 7 we explained now to describe a non-intersecting system of co-axal circles. In the figure of that theorem we have

$$OC^2 - CP^2 = OP^2 = OB^2,$$

which is constant for all positions of C; it follows that OC can never be less than OB, and that as C approaches B, the radius CP diminishes continuously, becoming zero when C coincides with B; while as C recedes from B, the radius CP increases, becoming infinite when C moves to infinity; hence we get a series of circles one outside the other, increasing in size from the point at B to the circle of infinite radius ultimately coinciding with

the radical axis OT. The point B is called a LIMITING POINT of the system, which is clearly symmetrical with



respect to the radical axis, the other extremity A of the diameter AOB being the other limiting point.

In a system of intersecting co-axal circles, the circles all have the same common chord, and there are no limiting points (or, as it is more usual to say, the limiting points are imaginary), for the smallest circle of the system has for diameter the common chord of all the other circles.

Theorem 8

The circle on a common tangent to two of a system of co-axal circles as diameter meets the line of centres in the limiting points, and these points are inverse to each other with respect to each circle of the system.

Let the common tangent to two of the circles meet their radical axis in T (fig. of Theorem 7), then T will be its midpoint, and $\therefore$ the centre of the circle described on it. And, as in the last theorem, the square on the radius $= OT^2 + OB^2 = TB^2$; $\therefore$ the circle passes through B, and also through A, the two limiting points of the system. Also $CA \cdot CB = CP^2$, and $\therefore$ A, B are inverse points with respect to the circle whose centre is C, and so for every other circle of the system.

Theorem 9

The radical axes of a system of three circles are concurrent.

Let A, B, C be the three circles, and let the radical axes of A and B and of A and C meet in T; then the tangents drawn from T to the circles B and C are each equal to the tangent from T to A, and $\therefore$ T lies on the radical axis of B and C.

DEF.—The point of concurrence of the three radical axes of a system of three circles is called the RADICAL CENTRE of the circles.

Theorem 10

The radical axes of a fixed circle and each of a system of co-axal circles are concurrent.

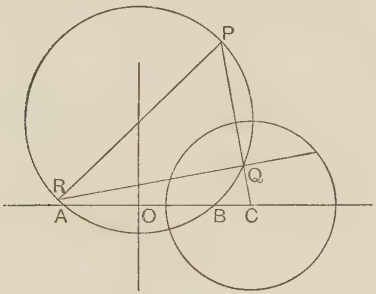
Let O be the fixed circle, and A, B, C, etc., the co-axal circles, and let the radical axis of these latter

circles meet the radical axis of O and A in T; then the tangent from T to O is equal to that from T to A, and ∴ to that from T to B, T to C, etc., and so the radical axis of O and any one of the other circles passes through T.

Theorem 11

The polars of a fixed point with regard to a system of co-axal circles are concurrent.

Let P be the fixed point, and Q its inverse point with regard to the circle whose centre is C. Let A



and B be the limiting points of the co-axal system, then A and B are inverse with regard to the circle whose centre is C;

∴ CP . CQ = CA . CB;

∴ A, B, Q, P are concyclic;

but if P is a fixed point, PAB is a fixed circle, and if PR is a diameter, R is a fixed point, and PQR being a right angle, QR is the polar of P with regard to the

circle whose centre is C; in like manner the polars of P with regard to all other circles of the system pass through the fixed point R.

NOTE.—If the limiting points are imaginary the circle PQR will still be fixed, namely, the circle through P, and orthogonal to the system of co-axal circles.

Theorem 12

If there are two systems of co-axal circles, one of which passes through two fixed points, and the other has these two fixed points for its limiting points, then any circle of either system is orthogonal to all the circles of the other system.

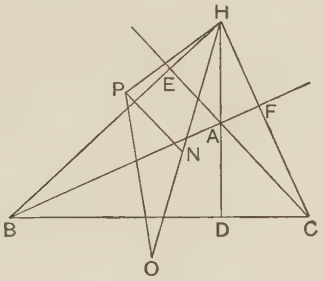
It is shown in Theorem 7 that the length of the tangent drawn from any point T to any one of the circles whose limiting points are A and B is equal to TA; consequently the circle through A and B whose centre is T—that is, every circle through A and B—is orthogonal to the other system of circles.

Theorem 13

The circumcircle of a triangle, the nine-point circle, and the circle to which the triangle is self-conjugate are co-axal.

With the notation of Chap. II., remembering that for the circle to which the triangle is self-conjugate to be real, the triangle must be obtuse-angled, let P be one of the points of intersection of the circumcircle and the circle to which the triangle is self-conjugate.

Then, since N is the midpoint of OH (Chap. II. Theorem 27, Cor.),



$$PH^2 + PO^2 = 2PN^2 + \tfrac{1}{2}OH^2;$$

that is, $HA \cdot HD + R^2 = 2PN^2 + \tfrac{1}{2}(R^2 + 2HA \cdot HD)$ (Chap. V. Theorem 9, and II. 31, taking cognisance of the sense of the lines);

$$\therefore \tfrac{1}{2}R^2 = 2PN^2, \quad \text{or } PN^2 = \tfrac{1}{4}R^2;$$

$\therefore$ P is on the nine-point circle (II. 27, Cor.)

CENTRES OF SIMILITUDE

Theorem 14

The straight lines joining the extremities of two parallel diameters of two given circles intersect in two fixed points, which divide the line joining the centres of the circles internally and externally into segments which have the same ratio as the radii of the circles.

Let ACB, A′C′B′ be parallel diameters of the two

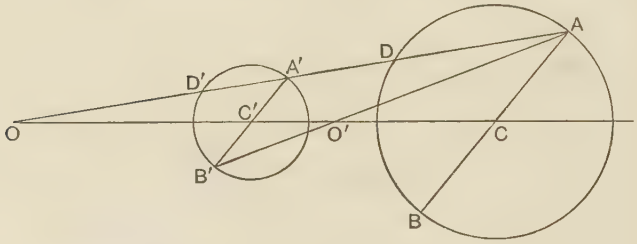
circles, and let AA′, AB′ meet the join of the centres CC′ in O, O′.

Then the triangles ACO, A′C′O are similar ;

$$\therefore \text{CO} : \text{C}'\text{O} = \text{CA} : \text{C}'\text{A}'.$$

Similarly $\text{CO}' : \text{C}'\text{O}' = \text{CA} : \text{C}'\text{B}'.$

In the same manner it may be shown that BB′,



BA′ pass through O, O′ respectively ; and so for any other parallel diameters.

Def.—These two fixed points are called respectively the internal and external centres of similitude of the two circles.

Theorem 15

The common tangents of two circles pass through one or other of the centres of similitude.

This is obvious from the previous proposition, as the diameters through the points of contact of a common tangent are perpendicular to it, and ∴ parallel.

Theorem 16

Any line drawn through two circles and one of their centres of similitude is cut by the circles in four points, which are two and two at the extremities of parallel radii; and the portion of the line between either pair of parallel radii is divided by the centre of similitude into segments which have the same ratio as the radii.

Let OD′A′DA (see last fig.) be drawn through one of the centres of similitude O, then in the two triangles OCA, OC′A′,

$$OC : CA = OC' : C'A',$$

and the angle at O common; ∴ the angles OCA, OC′A′ are either equal or supplementary.

In the case of equality it follows that CA and C′A′ are parallel, and also that

$$OA : OA' = CA : C'A'.$$

Theorem 17

If through a centre of similitude of two circles two straight lines be drawn, meeting one circle in R, R′, S, S′, *and the other in* r, r′, s, s′; *then the pairs of chords* RS, rs; R′S′, r′s′; RS′, rs′; R′S, r′s *are parallel; and the pairs* RS, r′s′; R′S′, rs; RS′, r′s; R′S, rs′ *intersect on the radical axis of the circles.*

Since OR : O*r* = OS : O*s* (Theorem 16), the triangles ORS, O*rs* are similar, and ∴ RS is parallel to *rs*; and so for the others.

Again, from above $OR \cdot Os = Or \cdot OS$;

also $Os \cdot Os' = Or \cdot Or'$;

$\therefore OR : Os' = OS : Or'$,

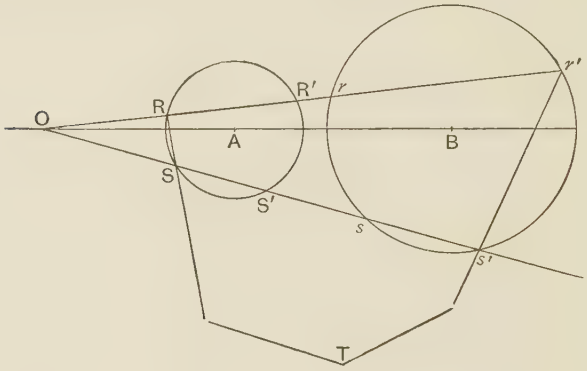
or $OR \cdot Or' = OS \cdot Os'$;

$\therefore$ R, S, r', s' are concyclic;

$\therefore$ if RS, $r's'$ meet in T,

$$TR \cdot TS = Tr' \cdot Ts',$$

and $\therefore$ T is on the radical axis of the two circles. Similarly for the other pairs of lines.



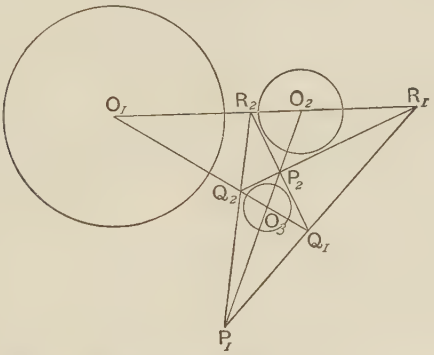
Theorem 18

The centres of similitude of three circles are the six vertices of a complete quadrilateral, of which the centres of the circles are the angular points of the diagonal triangle.

Let O_1, O_2, O_3 be the centres, r_1, r_2, r_3 the radii of the three circles,

P_1, P_2 the external and internal centres of similitude of the circles O_2, O_3,

Q_1, Q_2 the external and internal centres of similitude of the circles O_3, O_1,



R_1, R_2 the external and internal centres of similitude of the circles O_1, O_2.

Then we have to prove that $P_1Q_1R_1$, $P_1Q_2R_2$, $P_2Q_1R_2$, $P_2Q_2R_1$ are collinear. We have

$$\frac{O_2P_1}{O_3P_1}=\frac{r_2}{r_3},\quad \frac{O_3Q_1}{O_1Q_1}=\frac{r_3}{r_1},\quad \frac{O_1R_1}{O_2R_1}=\frac{r_1}{r_2};$$

$$\therefore O_2P_1 \cdot O_3Q_1 \cdot O_1R_1 = O_3P_1 \cdot O_1Q_1 \cdot O_2R_1 = -P_1O_3 \cdot Q_1O_1 \cdot R_1O_2;$$

$\therefore$ P_1, Q_1, R_1 are collinear (Chap. II., Theorem 8); and similarly for the others.

DEFS.—The four straight lines on which the six centres of similitude of three circles lie are called AXES OF SIMILITUDE. The circle described on the line join-

ing the centres of similitude of two circles as diameter is called the CIRCLE OF SIMILITUDE.

Theorem 19

The three circles of similitude of three given circles are co-axal.

With the notation of last theorem, let A, B be the points of intersection of the circles of similitude on P_1P_2 and Q_1Q_2; then, since O_2, O_3 are harmonic conjugates of P_1, P_2,

$$AO_2 : AO_3 = P_1O_2 : P_1O_3$$
$$= r_2 : r_3 \text{ (Chap. V. Theorem 1).}$$

Similarly $AO_3 : AO_1 = r_3 : r_1$;

$\therefore AO_2 : AO_1 = r_2 : r_1$;

$\therefore$ A lies on the circle of similitude on R_1R_2 (Chap. V. Theorem 2).

EXAMPLES ON CHAPTER VI

1. If two circles cut orthogonally at P, the lines joining P to the extremities of the common diameter pass through the extremities of the perpendicular diameters.

2. If P, P′ and Q, Q′ are pairs of inverse points with respect to a circle, the circle through P, P′, Q, Q′ is orthogonal to the circle of inversion.

3. Any circle having for diameter the distance between two conjugate points with respect to a circle is orthogonal to that circle.

4. The circle with respect to which a triangle is self-

conjugate is orthogonal to the circles described on the sides of the triangle as diameters.

5. If H is the orthocentre of the triangle ABC, the circles described on AB and CH as diameters are orthogonal.

6. Two circles whose centres are A and B cut orthogonally at C. BPQ cuts the circle A in P, Q, and CP, CQ cut the circle B in R, S; prove that RS is at right angles to BQ.

7. E is any point outside a circle, and F is a point on the polar of E; prove that the circles whose centres are E and F, and which cut the given circle orthogonally, are orthogonal.

8. If A and B be two fixed points on a fixed circle whose centre is C, and QCR be any diameter, the circles ACQ, BCR will intersect on the circle which cuts the fixed circle orthogonally at A and B.

9. If AD, BE, CF are the altitudes and H the orthocentre of the triangle ABC, the circles with centres A, B, C and radii whose squares are AH.AD, BH.BE, CH.CF are orthogonal.

10. Describe a circle which shall have a given radius and cut two given circles orthogonally.

11. Given two points P and Q, find the locus of the centre of a circle the tangents to which from P and Q shall be equal to given lines.

12. Show that the middle points of the four common tangents to two circles which lie outside each other are collinear.

13. The tangents to every circle of a co-axal system drawn from a limiting point are bisected by the radical axis.

14. A straight line is the radical axis of its inverse and the circle of inversion.

15. When two circles intersect, the circle through their common points and the centre of inversion inverts into the radical axis of the two inverse circles.

16. The difference of the squares on the tangents from *any* point to two circles is equal to twice the rectangle under the join of their centres and the distance of the point from the radical axis.

17. Prove that the circles whose radii are r_1, r_2, r_3, and whose centres are the three collinear points O_1, O_2, O_3, will be co-axal if

$$r_1^2 . O_2O_3 + r_2^2 . O_3O_1 + r_3^2 . O_1O_2 + O_2O_3 . O_3O_1 . O_1O_2 = 0.$$

18. If A, B, C be centres of three co-axal circles, and PR, PS, PT tangents to them from any point P, then

$$BC . PR^2 + CA . PS^2 + AB . PT^2 = 0.$$

19. The squares on the tangents to two circles from any point on a co-axal circle are proportional to the distances of their centres from the centre of the third circle.

20. A variable circle cuts two fixed circles orthogonally ; find the locus of its centre.

21. If any straight line PQRS be drawn cutting two non-intersecting circles in P, S, and Q, R respectively, the segments PQ, RS subtend equal angles at either limiting point.

22. If D, E, F be the feet of the altitudes of the triangle ABC, and BC and EF meet in X, CA and FD in Y, AB and DE in Z, then X, Y, Z are on the radical axis of the circum- and nine-point circles.

23. If X, Y, Z be collinear points on the sides BC, CA, AB of a triangle, the circles on AX, BY, CZ as diameters are co-axal.

24. The circles on the diagonals of a quadrilateral as diameters are co-axal.

25. If circles are described on the three sides of a triangle as diameters, their radical centre is the orthocentre of the triangle.

26. If a given point and straight line be pole and polar with respect to every circle of a system, the system is co-axal.

27. Prove that there are two points at which three circles subtend the same angle, and that their join passes through the radical centre of the circles.

28. If an intersecting co-axal system is inverted with respect to either common point as centre of inversion, the inverse is a system of concurrent straight lines.

29. If a non-intersecting co-axal system is inverted with respect to either limiting point, the inverse is a system of concentric circles, having the other limiting point for centre.

30. Invert the theorem that all radii of concentric circles cut them orthogonally.

31. Two circles, inverses of each other, are co-axal with the circle of inversion.

32. T is a point on the radical axis of two non-intersecting circles, the straight line through their centres meets them in A, B and C, D respectively; if TA, TB meet the first circle in P, p respectively, and TC, TD meet the second circle in Q, q respectively, prove that T, q, Q, P, p are concyclic.

33. Two fixed circles touch at A, and P, P′ are points one on each circle, such that PAP′ is a right angle; show that PP′ passes through a fixed point.

34. Two parallel chords of two circles are drawn proportional to their radii; show that the straight lines joining their extremities pass through the centres of similitude.

35. If two circles are in contact, corresponding extremities of parallel diameters are collinear with the point of contact.

36. Two circles touch externally at A; S is their external centre of similitude and SCB a common tangent; show that $SA^2 = SB \cdot SC$.

37. A is the internal centre of similitude of two circles whose centres are C and C′, an external common tangent touches the circles in D and D′, and AN is perpendicular to DD′. Prove that CD′ and C′D intersect in the middle point of AN.

38. Show that if two circles are given in a plane a point can always be found such that the chords cut off any straight line through the point by the two circles are in a constant ratio.

39. The two poles of any line through a centre of similitude of two circles are collinear with that centre.

40. Every two lines through a centre of similitude of two circles which are conjugate lines with respect to one of the circles are also conjugate with respect to the other.

41. The circle of similitude of two circles is co-axal with them.

42. Show that the orthocentre of a triangle is a centre of similitude of the nine-point and circum- circles; find the other centre of similitude.

43. If tangents be drawn to a pair of circles from any point on their circle of similitude, the ratio of the lengths of the tangents is constant.

44. If C is the centre of the circle of similitude of two circles whose centres are O, O′ and radii R, R′ respectively, then

$$OC : O'C = R^2 : R'^2.$$

45. If a variable circle touches two fixed circles, the line joining the points of contact goes through a centre of similitude of the fixed circles.

46. If TT′ is a common tangent and PQpq a secant of two circles passing through the same centre of similitude, so that P, p and Q, q are corresponding points, then

$$TT'^2 = P_p \cdot Q_q.$$

47. If a system of co-axal circles having real limiting points L, L′ be inverted from any point E in their line of centres, the radius of the circle of inversion being a mean proportional between EL and EL′, then each circle of the system inverts into another circle of the system, and the circle which passes through the centre of inversion inverts into the radical axis of the system, and is the circle of similitude of each pair of circles which invert into each other. (H. D. Ellis, M.A.)

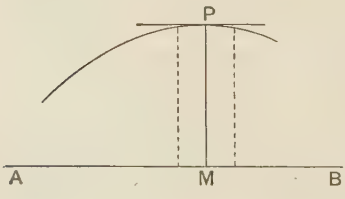
CHAPTER VII

GEOMETRICAL MAXIMA AND MINIMA

Definitions, etc.

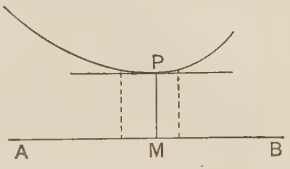
WHEN any geometrical magnitude, such as a straight line, angle, or area, varies subject to certain prescribed conditions, its greatest possible value is called the MAXIMUM and its least possible value the MINIMUM.

It will be seen that if the magnitude under consideration varies *continuously*, that is, changes gradually and without any break in its series of values, then on either side of the maximum or minimum value there will be two equal values; if a maximum, both a little less than the maximum; if a minimum, both a little greater than the minimum.



For, if the magnitude be the length of the perpendicular from a point on a given curve to a given straight line, it is evident from the figure that, if the curve be concave to the given straight

line AB, the greatest value of the perpendicular PM is when the tangent to the curve at P is parallel to the given straight line AB, and that on either side of this position of PM and near to it there are positions for which the lengths of the perpendicular are the same and a little less than the maximum. In like manner, if the curve be convex to the given straight line, then the least value of the perpendicular PM is when the tangent to the curve at P is parallel to AB, and near to the minimum and on either side of it are two positions for which the lengths of the perpendicular are equal, and differ but slightly from the minimum.



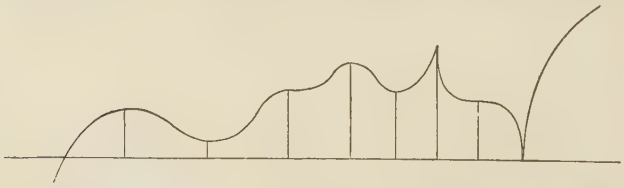
And generally, if M denote the magnitude and M_A its maximum value; as M increases gradually suppose it passes through a value M′ just before reaching M_A, then after reaching M_A the magnitude M decreases continuously, and hence will pass again through the value M′, so that on either side of the maximum M_A there are two equal values M′ of the magnitude, and by taking M′ nearer and nearer to M_A these two equal values of M will ultimately coincide with M_A. Similarly for the minimum.

This is sometimes expressed by saying that the maximum and minimum values are *stationary* values; for by taking positions of the variable magnitude near to and on either side of the maximum or minimum we get equal values for the magnitude, or the

magnitude is stationary in value near a maximum or minimum.

This principle of stationary values is often of use in solving problems on maxima and minima; for we can take two positions of the magnitude near to one another and assume them to give equal values to the magnitude, and then from the equality of the parts of the figures, and by making the two chosen positions ultimately coincident, we can deduce what the correct position of the maximum or minimum must be.

It should, however, be observed that the determination of the stationary values of a varying magnitude does not necessarily give maxima or minima values: thus, in the problem just considered of finding the greatest or least perpendicular from a given curve to a given straight line, we have seen that the required perpendicular is drawn from a point on the curve at which the tangent is parallel to the given line. If, however, the parallel tangent also cut the curve at that point, then the perpendicular, although it has a stationary value, would not be a maximum or minimum, as may be seen by the subjoined figure, where other



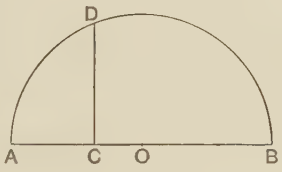
exceptional cases are suggested. The complete consideration of maxima and minima requires the aid of

Differential Calculus, and is beyond our present province.

Theorem 1

The rectangle of the segments of a given straight line is a maximum, and the sum of the squares on them a minimum, when the segments are equal.

Let AB be the given straight line, C any point in it. Describe a semicircle on AB, and draw CD at right angles to AB.



Then, since the rectangle $AC \cdot CB = CD^2$, the rectangle will obviously be a maximum when C is the centre of the circle, that is, when the segments are equal.

Again, $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$;

$\therefore$ $AC^2 + CB^2$ will be a minimum when the rectangle $AC \cdot CB$ is a maximum.

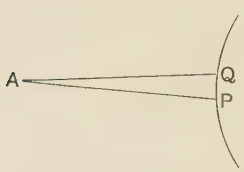
Theorem 2

The maximum or minimum straight line from a given point to a circle is the normal through the point.

This is proved in Euclid III. 7 and 8, the line being a maximum when drawn to the concave, and a minimum when drawn to the convex portion of the circumference.

The following proof applies to any curve continuously convex or concave :—

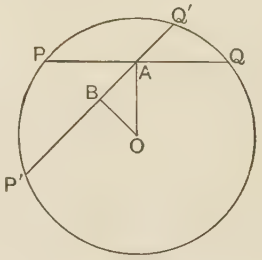
Let AP be the minimum line drawn from A, and AQ a consecutive position. Then in the limit AP = AQ, ∴ the triangle APQ is isosceles. And since the angle PAQ is indefinitely small, each of the angles APQ, AQP is ultimately a right angle. And PQ being in the limit the direction of the tangent at P, AP is normal at P. Similarly, if the curve were concave to A, the normal AP would be a maximum.



Theorem 3

Given a circle and a point within it, the chord which is bisected by the point (1) *is a minimum,* (2) *cuts off a minimum area.*

Let A be the given point, PQ the chord bisected at A, P′Q′ any other chord through A. Join A to the centre O, then OA is perpendicular to PQ and draw OB perpendicular to P′Q′.



Then the angle OAB is < the right angle OBA ;

∴ OB is < OA ; and the chords which are nearer the centre are greater than those more remote ; ∴ P′Q′ is >

PQ ; ∴ PQ is the minimum chord through A. And it is evident that the chord which is more remote from the centre cuts off the smaller segment, ∴ the segment PQ′Q is a minimum ; and of course the segment PP′Q, which is greater than a semicircle, is a maximum.

Theorem 4

Given two straight lines and a point within the angle formed by them, of all straight lines drawn through the point that which is bisected by it cuts off the minimum triangle.

Let AB, AC be the two lines, and O the point ; and let POQ cut off the minimum triangle.

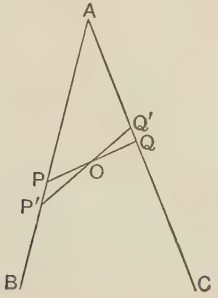
Draw through O a consecutive line P′OQ′. Then in the limit the triangle PAQ = P′AQ′ ;

∴ triangle POP′ = QOQ′ ;

∴ the rectangle OP . OP′ = OQ . OQ′ ;

ultimately, that is, $OP^2 = OQ^2$,

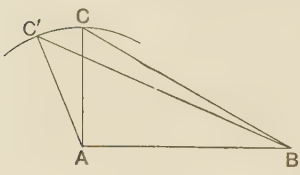
or OP = OQ.



Theorem 5

Given two sides of a triangle in magnitude, the area is a maximum when they contain a right angle.

Let AB be one of the given sides, and suppose it fixed. Then of all triangles described on AB with the adjacent side AC given, that one will be greatest which has the greatest altitude, and this evidently will be that triangle whose altitude is AC; that is, of which CAB is a right angle.



Theorem 6

Of all triangles on the same base the isosceles has—

(1) *Minimum perimeter when the area is given.*

(2) *Maximum area when the perimeter is given.*

(1) Let AB be the given base; then the vertices of all triangles of given area described on AB will lie on a straight line CD parallel to AB.

Draw BE perpendicular to CD and produce it to F, making EF = BE.

Join AF, meeting CD in P, and from any other point Q in CD, draw QA, QB, QF.

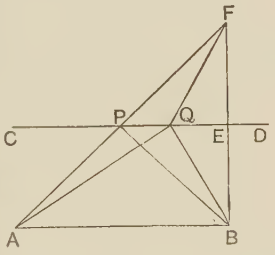
It is obvious that BP = FP, and BQ = FQ;

also $AF < AQ + QF$;

that is, $AP + PB < AQ + QB$,

or APB has the minimum perimeter; and since AP = PF = PB, this triangle is isosceles.

(2) The locus of the vertices of all triangles on AB which have a given perimeter is an ellipse, whose foci are A, B, and the greatest triangle will obviously have its vertex at the extremity of the minor axis, and so will be isosceles.



Or (2) may be deduced from (1) as follows: — If APB be the triangle of maximum area, and if AQB be a consecutive position, then area APB = AQB. Also AP + PB = AQ + QB. But these are the same two conditions which are required to determine the minimum perimeter when the area is given, and from (1) we have seen that this is when the triangle APB is isosceles.

Theorem 7

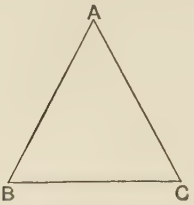
Of all triangles the equilateral has—

(1) *Minimum perimeter when the area is given.*
(2) *Maximum area when the perimeter is given.*

(1) Suppose one side BC to remain fixed. Then by Theorem 6 we see that the perimeter will be a minimum when AB = AC.

Similarly, if we consider AB to be fixed, we shall have AC = CB;

$\therefore$ the equilateral triangle has the minimum perimeter.



(2) may be shown in like manner.

Theorem 8

Of all polygons of the same number of sides the equilateral polygon has—

(1) *Minimum perimeter when the area is given.*
(2) *Maximum area when the perimeter is given.*

(1) As in Theorem 7, suppose all the angular points except one to remain fixed. Then we see that the perimeter of the polygon is a minimum when the two sides terminated in this point are equal. Similarly every pair of adjacent sides must be equal; that is, the polygon must be equilateral.

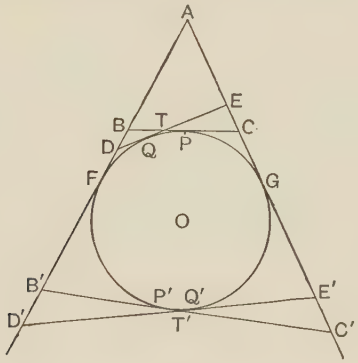
(2) may be shown in like manner.

Theorem 9

If a tangent to a circle cut two other tangents drawn from a fixed point, that tangent so drawn will be a minimum

which is bisected at its point of contact, and will cut off a maximum or a minimum triangle according as it excludes or includes the given circle.

Let AF, AG be the two fixed tangents, BPC the tangent at P which cuts off the maximum triangle ABC,



DQE a consecutive tangent, and let the two tangents meet in T.

Then ultimately the triangle

$$ADE = ABC;$$
$$\therefore BTD = CTE;$$
$$\therefore \text{rectangle } TB.TD = TC.TE.$$

But when Q moves up and coincides with P, so also will T, and consequently in the limit we have

$$PB^2 = PC^2, \text{ or } PB = PC.$$

An identical proof will establish the theorem for the

tangent B′P′C′ drawn to the circle on the side remote from A. It is also obvious that the triangle ABC is a maximum and AB′C′ a minimum.

Again, using the notation of Chap. II. for the triangle ABC, we have

$$\Delta = r_1(s - a) \text{ (Chap. II. Theorem 13),}$$

or $r_1 a = r_1 s - \Delta.$

Now $s = \text{AF} = \text{AG}$ (Chap. II. Theorem 12),

and is $\therefore$ constant, as also is r_1; and consequently a is a minimum when Δ is a maximum.

So for the triangle AB′C′, if Δ' be its area, s' its semiperimeter, and $\text{B}'\text{C}' = a'$, we have

$$\Delta' = r_1 s' = r_1(s' - a' + a');$$

$$\therefore r_1 a' = \Delta' - r_1(s' - a'),$$

and since $s' - a' = \text{AF}$, a' is a minimum when Δ' is a minimum.

Theorem 10

Of all polygons of a given number of sides circumscribing a circle, that which is regular is a minimum both in perimeter and area.

Let all the sides of the polygon be fixed but one, BC (see last fig.) Then the fixed area OFAG is made up of OFBCG and the triangle ABC;

$\therefore$ OFBCG is a minimum when ABC is a maximum, that is, when BC is bisected at P; and at the same time BC, and $\therefore$ FBCG, is a minimum.

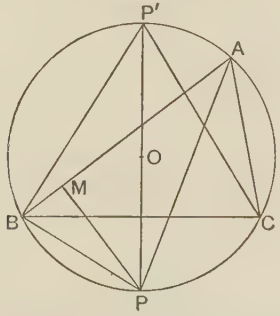
In like manner we may show that each side of the

polygon must be bisected at its point of contact in order to make either the area or the perimeter a minimum. And under these circumstances it is evident that the polygon must be equilateral, and that each side must subtend the same angle at the centre of the circle; that is, the polygon must be regular.

Theorem 11

Of all triangles inscribed in a circle on a given chord as base, the isosceles triangle is a maximum both in perimeter and area.

Using the notation of Chap. II., it is obvious that the isosceles triangle P′BC is greater than any other triangle ABC, as its altitude is greater.



Draw PM perpendicular to AB. Then the triangles APM, P′PB are similar, and the diameter P′P is > AP ;

$\therefore$ P′B is > AM.

But $2AM = AB + CA$ (Chap. II. Theorem 17);

$\therefore$ P′BC has the maximum perimeter.

Theorem 12

If two convex polygons lie on the same base, and the vertices of one lie upon the sides or within the area of the other, this other has the greater perimeter.

If the vertices of the inner polygon lie on the sides of the other, the theorem is obvious from the fact that two sides of a triangle are greater than the third. And this proposition, together with "If from the ends of a side of a triangle there be drawn two straight lines to a point within the triangle these two straight lines are less than the other two sides of the triangle," are sufficient to establish the theorem in all cases.

Theorem 13

Of all polygons of a given number of sides inscribed in a circle, that which is regular is a maximum both in perimeter and area.

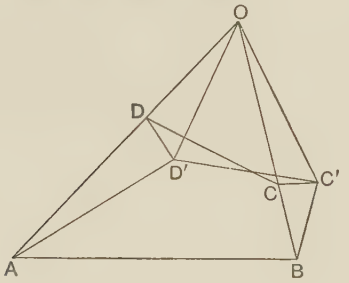
This follows at once from Theorem 11 by a similar method to that adopted in Theorems 8 and 10.

Theorem 14

Of all quadrilaterals which can be formed from four straight lines of given lengths, the maximum is that which can be inscribed in a circle.

Let ABCD be the position of maximum area. Take ABC′D′ a consecutive position, keeping AB fixed. Let AD, BC meet in O. Then, since AD = AD′,

$\therefore$ ultimately the angle ADD′ is a right angle;
$\therefore$ also ODD′ is a right angle, and OD = OD′ ultimately.



Similarly OC = OC′ ultimately.

And CD = C′D′;

$\therefore$ angle DOC = D′OC′,

and triangle OCD = OC′D′;

$\therefore$ angle DOD′ = COC′.

Again, in the limit the area ABCD = ABC′D′;

$\therefore$ triangle OAB = area OC′BAD′.

Taking away the common part OBAD′, we get the triangle OAD′ = OBC′, and an angle AOD′ of the one = BOC′ of the other;

$\therefore$ OA . OD′ = OB . OC′;

$\therefore$ in the limit OA . OD = OB . OC;

$\therefore$ A, B, C, D are concyclic.

Cor. 1.—Of all polygons which can be formed from straight lines which are given both in number and length, the maximum is that about which a circle can be described.

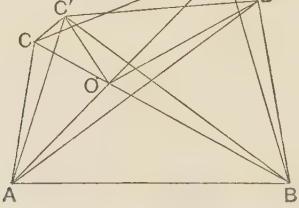
This is evident from the theorem, as every four consecutive vertices of the polygon must be concyclic, and as three points fix a circle, all the vertices must lie on the same circle.

Cor. 2.—If we suppose the number of sides to be indefinitely great and the length of each indefinitely diminished, the maximum polygon will ultimately coincide with the circle which circumscribes it. Hence we see that the area of a circle is greater than that of any other closed figure having the same perimeter.

Theorem 15

If two of the lines in Theorem 14 *be crossed, the difference of the areas of the triangles so formed is a minimum when the vertices are concyclic.*

Let AB, BC, CD, DA be the position occupied by the four given lines when the difference of the areas of



AOB and COD is a minimum. Let AB, BC′, C′D′, D′A be a consecutive position. Then it may be shown, as in

Theorem 14, that the triangle COD = C′OD′, and the angle COC′ = DOD′; and consequently angle BOC′ = AOD′.

The triangle AOB – triangle COD,

which = ABC – ADC,

is ultimately = ABC′ – AD′C′

= ABO + AOC′ + C′OB – (AOD′ + AOC′ + C′OD′):

hence, remembering that COD = C′OD′, we get

$$0 = C'OB - AOD',$$

or $$AOD' = C'OB;$$

and as the angle $$AOD' = C'OB,$$ it follows that

$$AO \cdot OD' = BO \cdot OC',$$

and in the limit $$AO \cdot OD = BO \cdot OC;$$

∴ A, B, C, D are concyclic.

It is evident that this position gives a minimum and not a maximum, from the consideration of the case when C lies on AD.

Theorem 16

The sum of the squares of the distances of a point from the angular points of a triangle is a minimum when the point is the centroid.

Let P be the point. Join P to the centroid G, and draw Aa, Bb, Cc, A′a′ perpendicular to PG. Then

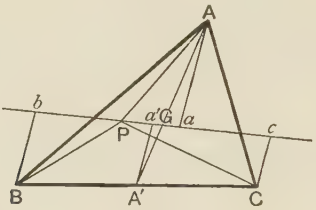
$$AP^2 = AG^2 + PG^2 + 2PG \cdot Ga$$
$$BP^2 = BG^2 + PG^2 - 2PG \cdot bG$$
$$CP^2 = CG^2 + PG^2 + 2PG \cdot Gc;$$

but $AG = 2GA'$: $\therefore\ Ga = 2a'G$

$Gb = -a'G + a'b$

$Gc = -a'G + a'c,$

and $BA' = A'C$; $\therefore\ a'b = -a'c$;



$\therefore\ Ga + Gb + Gc = 0$;

$\therefore\ AP^2 + BP^2 + CP^2 = AG^2 + BG^2 + CG^2 + 3PG^2,$

and is a minimum when P coincides with G.

Theorem 17

The sum of the squares of the distances of a point from the sides of a triangle is a minimum when the point is the symmedian point.

Let S be a point at a fixed distance from the side BC, and let SL, SM, SN be perpendicular to BC, CA, AB respectively. Then

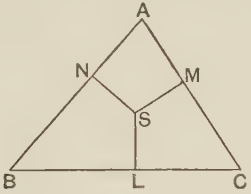
$$(b^2 + c^2)(SM^2 + SN^2) = (b \,.\, SM + c \,.\, SN)^2 + (c \,.\, SM - b \,.\, SN)^2 ;$$

and since $a \,.\, SL + b \,.\, SM + c \,.\, SN = 2\Delta$, and SL is constant,

$\therefore\ SM^2 + SN^2$ is a minimum when $c \,.\, SM - b \,.\, SN = 0$.

$\therefore$ S must lie on the symmedian AK (Chap. II. Theorem

36). Hence the sum of the squares of the perpendiculars from the symmedian point K on the three sides must be less than from any other point.



Examples on Chapter VII

1. The minimum distance between two circles is the portion intercepted on the join of their centres.

2. When all the sides except one of a rectilinear figure are given, the area is a maximum when the figure can be inscribed in a circle of which the side not given is a diameter.

3. Of all right-angled triangles on the same hypotenuse, the isosceles has the maximum perimeter.

4. Find a point in a given straight line such that the sum of the squares of its distances from two given points may be a minimum.

5. A and B are two given points on the same side of a given straight line; find the point on the straight line at which AB subtends the maximum angle.

6. A and B are two given points outside a given circle; find the points on the circle at which AB subtends the maximum and minimum angles.

7. For the point of bisection of any side of a triangle the area of the inscribed parallelogram formed by drawing parallels to the sides is a maximum.

8. AB is a chord of a circle and P any point on the circumference ; show that $AP^2 + BP^2$ is a maximum or minimum according as P bisects the major or minor arc cut off by AB.

9. A and B are fixed points on the same side of a given straight line XY, P is a point in XY ; show that the difference between AP and BP is a maximum when A, B, P are collinear, and their sum a minimum when XY bisects the exterior angle APB.

10. A and B are fixed points on opposite sides of a given straight line XY, P is a point in XY ; show that the sum of AP and BP is a minimum when A, P, B are collinear, and their difference a maximum when XY bisects the angle APB.

11. In a given triangle inscribe the triangle of minimum perimeter.

12. The feet of the perpendiculars from the intersection of the diagonals on the sides of a cyclic quadrilateral P are joined to form a second quadrilateral Q ; show that Q is a quadrilateral of minimum perimeter inscribed in P.

13. A and B are fixed points on the same side of a circular arc, P is a point on the arc ; show that AP + BP is a minimum or maximum when the radius through P bisects the angle APB, according as the arc is convex or concave to AB.

14. In a given triangle which has each of its angles less than two-thirds of two right angles, show that the point the sum of whose distances from the angular points of the triangle is a minimum is such that the three sides of the triangle subtend at it equal angles.

15. Of all lines through either point of intersection of two circles, that whose segment intercepted between the

circles is of maximum length, and subtends at the other point of intersection the triangle of maximum area is that which is perpendicular to the common chord.

16. Of all lines through a fixed point, that whose segments intercepted in opposite directions between the point and two fixed lines contain the rectangle of minimum area is that which makes equal angles with the lines.

17. Of all lines through either point of intersection of two circles, that whose segments intercepted in opposite directions between the point and circles contain the rectangle of maximum area is that which makes equal angles with the circles.

18. A and B are fixed points on a circle, PQ is a fixed chord; find a point C on the circumference so that the part of PQ intercepted by CA, CB shall be a maximum.

19. Bisect a triangle by the shortest possible straight line.

20. Having given the lengths of the sides of a quadrilateral, when will the rectangle under its diagonals be a maximum?

21. A and B are two fixed points, and P is a variable point on a fixed straight line not passing through A or B; determine the positions of P for which the ratio PA : PB is a maximum or minimum.

22. A variable chord AB of a curve has an envelope which it touches at C; show that when AB is a maximum or minimum the normals at A, B, and C are concurrent.

CHAPTER VIII

CROSS RATIOS, INVOLUTION, AND RECIPROCAL POLARS

Definitions

A SYSTEM of collinear points is called a RANGE; the straight line in which the points lie is called the BASE of the range; each of the collinear points is called a POINT or ELEMENT of the range. A system of concurrent straight lines is called a PENCIL; the point where the lines meet is called the VERTEX or FOCUS of the pencil; each of the concurrent lines is called a RAY or ELEMENT of the pencil. If a range consists of four points A, B, C, D, the ratio $\frac{AC}{BC} : \frac{AD}{BD}$ is called the CROSS RATIO or ANHARMONIC RATIO of the range, and is written (ABCD); the points A and B are considered in this definition to be conjugate points.

NOTE.—The cross or anharmonic ratio (ABCD) is defined in various ways by different writers, the most usual being $\frac{AB}{AD} : \frac{CB}{CD}$, or $\frac{AB \cdot CD}{AD \cdot CB}$. The definition we have given is that of Cremona, which has the advantages

of making A and B conjugate points, and also of fitting in readily with our notation in the chapter on Harmonic ranges and pencils. It will be observed that if $(ABCD) = -1$, then ACBD is a harmonic range.

If any two of the four letters A, B, C, D be interchanged, and we also interchange the other two, the value of the cross ratio is unaltered; or

$$(ABCD) = (BADC) = (CDAB) = (DCBA);$$

this is evident from writing down the values of the four cross ratios thus:

$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{BD \cdot AC}{AD \cdot BC} = \frac{CA \cdot DB}{DA \cdot CB} = \frac{DB \cdot CA}{CB \cdot DA}.$$

Since the four letters A, B, C, D may be arranged in twenty-four different ways, there are twenty-four cross ratios with the same four points; but, from above, in these twenty-four cases each ratio appears four times, and the twenty-four ratios are thus reduced to six different ratios, of which three will be seen to be the reciprocals of the other three.

Theorem 1

If the cross ratio $(ABCD) = \lambda$, *then the other five ratios derived from the same four collinear points are*

$$\frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda - 1}{\lambda}, \quad \frac{\lambda}{\lambda - 1}.$$

It is obvious from the definition that

$$(ABCD) = (BADC) = (CDAB) = (DCBA) = \lambda;$$

also that $(ABDC) = (BACD) = (CDBA) = (DCAB) = \frac{1}{\lambda}$.

Now $(ACBD) = \frac{AB \cdot CD}{CB \cdot AD} = \frac{(AC - BC)(BD - BC)}{CB \cdot AD}$

$$= \frac{AC \cdot BD - BC(BD - BC + AC)}{CB \cdot AD}$$

$$= -\frac{AC \cdot BD}{BC \cdot AD} - \frac{BC \cdot AD}{CB \cdot AD} = 1 - \lambda;$$

$\therefore (ACBD) = (BDAC) = (CADB) = (DBCA) = 1 - \lambda,$

and $(ACDB) = (BDCA) = (CABD) = (DBAC) = \frac{1}{1 - \lambda}$.

Again,

$$(ADBC) = \frac{AB}{DB} : \frac{AC}{DC} = -\left(\frac{AB}{CB} : \frac{AD}{CD}\right) \times \left(\frac{AD}{BD} : \frac{AC}{BC}\right)$$

$$= -(ACBD) \times (ABDC) = -\frac{1 - \lambda}{\lambda} = \frac{\lambda - 1}{\lambda};$$

$\therefore (ADBC) = (BCAD) = (CBDA) = (DACB) = \frac{\lambda - 1}{\lambda},$

and $(ADCB) = (BCDA) = (CBAD) = (DABC) = \frac{\lambda}{\lambda - 1}$.

Cor.—If $P(=\lambda)$, $Q\left(=\frac{1}{1 - \lambda}\right)$, $R\left(=\frac{\lambda - 1}{\lambda}\right)$ and their reciprocals denote the six cross ratios, then

$$P \cdot Q \cdot R = -1,$$

and

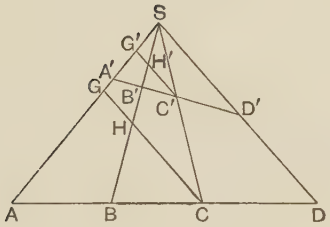
$$Q + \frac{1}{R} = R + \frac{1}{P} = P + \frac{1}{Q} = 1.$$

Theorem 2

If S *be a point without the range* ABCD, *and if through* C

a line be drawn parallel to SD *meeting* SA, SB *in* G, H *respectively, then* GC : HC = (ABCD).

For
$$\frac{AC \cdot BD}{BC \cdot AD} = \frac{AC}{AD} \cdot \frac{BD}{BC}$$
$$= \frac{GC}{SD} \cdot \frac{SD}{HC} \text{ by similar triangles}$$
$$= \frac{GC}{HC}.$$



Theorem 3

If in the figure of the last proposition any other transversal be drawn meeting SA, SB, SC, SD *in* A′, B′, C′, D′ *respectively, then the cross ratio* (A′B′C′D′) = (ABCD).

Draw C′H′G′ parallel to CHG or SD. Then
$$(A'B'C'D') = \frac{G'C'}{H'C'} = \frac{GC}{HC} = (ABCD).$$

NOTE.—We may speak of the cross or anharmonic ratio of a pencil SA, SB, SC, SD of four rays, meaning

thereby the cross ratio in any transversal cutting the pencil. The cross ratio of the pencil is written S(ABCD).

Theorem 4

Equiangular pencils have equal cross ratios.

This is obvious since, if one pencil be supposed to be placed on the other so that the vertices and a corresponding ray of each be coincident, then the other rays will coincide.

Cor.—If A, B, C, D be four fixed points and P a variable point on the circumference of a circle, the cross ratio P(ABCD) is invariable, for the angles APB, BPC, CPD are constant.

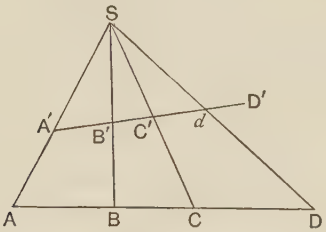
Def.—If two ranges or pencils have equal cross ratios, they are said to be EQUI-CROSS, or EQUI-ANHARMONIC. Such ranges or pencils are sometimes called HOMOGRAPHIC. This latter term is, however, more frequently used when the ranges or pencils consist of more than four elements, and are such that the cross ratio of any four elements of one range or pencil is equal to that of the corresponding four elements of the other range or pencil.

Theorem 5

If two equi-cross ranges be so placed that the three straight lines joining three pairs of corresponding elements are

concurrent, then the line joining the fourth pair will be concurrent with them.

Let ABCD, A′B′C′D′ be the ranges such that AA′,



BB′, CC′ meet in S. Join SD, cutting A′B′C′ in *d*. Then we have a pencil cut by two transversals;

$\therefore (A'B'C'd) = (ABCD) = (A'B'C'D')$;

$\therefore$ *d* and D′ are coincident;

$\therefore$ S, D′, D are collinear.

Cor.—The lines joining corresponding elements of two equi-cross ranges which have two corresponding elements coincident are concurrent.

Theorem 6

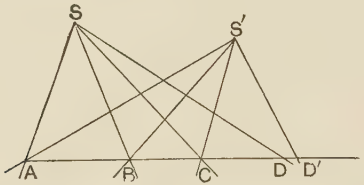
If two equi-cross pencils be so placed that the three points of intersection of three pairs of corresponding rays are in a straight line, the point of intersection of the fourth pair will be in the same straight line.

Let S and S′ be the vertices, and let the line ABC,

on which lie the three points of intersection, cut the fourth pair of rays in D and D′. Then

$$\begin{aligned}(ABCD) &= S(ABCD) \\ &= S'(ABCD') \\ &= (ABCD')\,;\end{aligned}$$

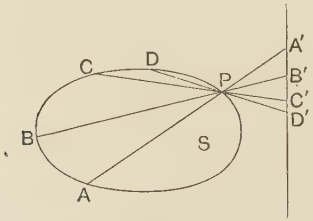
∴ D and D′ coincide, and the four points of intersection are collinear.



Cor.—The points of intersection of corresponding elements of two equi-cross pencils which have two corresponding rays coincident are collinear.

Theorem 7

Pencils whose rays pass through four fixed points on a conic, and whose vertices lie on the conic, are equi-cross.



Let the S-directrix cut the pencil P(ABCD) in

A′, B′, C′, D′. Then A′S bisects the exterior angle between SA and SP;

$$\therefore \text{ angle } A'SP = \tfrac{1}{2} \text{ suppt. } ASP,$$

similarly $B'SP =$ „ BSP;

$$\therefore A'SB' = \tfrac{1}{2}ASB, \text{ which is constant.}$$

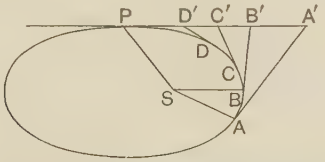
Similarly B′SC′ and C′SD′ are constant angles, and

$$P(ABCD) = P(A'B'C'D') = S(A'B'C'D') = \text{constant.}$$

Cor.—It is easily seen that if Q be any point not lying on the conic, then the cross ratio Q(ABCD) cannot be equal to P(ABCD). For if QA meet BC in A″ and the conic in R, then if Q(ABCD) = R(ABCD), A″, B, C, D would be collinear (6, Cor.), which is impossible. Hence it follows that the locus of a point at which four fixed points subtend a constant cross ratio is a conic. For if a conic[1] be described passing through the four fixed points and some one position of the moving point, then points on this conic, and only such points, satisfy the given condition.

Theorem 8

Ranges whose points lie on four fixed tangents to a conic, and whose bases touch the conic, are equi-cross.



[1] There is, of course, an incompleteness in this proof if it may not be assumed that a conic can be described through five given points, and this proposition is not proved in treatises on Geometrical Conics.

Let the tangent at P cut the four fixed tangents at A, B, C, D in A′, B′, C′, D′ respectively. Then

$$\text{angle } A'SP = \tfrac{1}{2}PSA \text{ and } B'SP = \tfrac{1}{2}PSB;$$

$$\therefore A'SB' = \tfrac{1}{2}ASB, \text{ which is constant.}$$

Similarly B′SC′ and C′SD′ are constant angles, and

$$(A'B'C'D') = S(A'B'C'D') = \text{constant.}$$

Cor.—Suppose any straight line which does not touch the conic to intercept on the four fixed tangents a range A″B″C″D″; then if $(A''B''C''D'') = (A'B'C'D')$, B′B″, C′C″, D′D″, three tangents to the conic, would be concurrent (5, Cor.), which is impossible.

Hence it follows that a variable straight line, on which four fixed straight lines intercept a range of constant cross ratio, must touch a fixed conic. For if a conic be described touching the four fixed lines and the moving line in any one position,[1] then tangents, and no other straight lines, satisfy the given condition.

NOTE ON HARMONIC RANGES

From the definition of a harmonic range it is evident that if $(ABCD) = -1$, the range ACBD is harmonic. If $(ABCD) = -1$, it follows that

[1] As in the corollary to Theorem 7, the proof is not complete if it may not be assumed that a conic can be drawn touching five given straight lines. We have not considered it necessary to specify the obvious restrictions to this and the previous corollary, that in the one case no three of the given points should be collinear, and in the other that no three of the given lines should be concurrent. The proofs of Theorems 7 and 8 are due to the late Mr. B. W. Horne, Fellow of St. John's College, Cambridge, and first appeared in the *Quarterly Journal of Mathematics*, vol. iv. p. 278.

$$(ABDC) = \frac{1}{(ABCD)} = -1,$$

and $$\therefore (ABDC) = (ABCD)$$

Or, if a range ACBD is harmonic, the points C and D (or A and B) are interchangeable without altering the value of the cross ratio of the range.

Conversely, if $(ABCD) = (ABDC)$, we have

$$(ABCD) = \frac{1}{(ABCD)}; \quad \therefore (ABCD) = \pm 1,$$

and it will be found that if we take $(ABCD) = +1$, some of the four points must coincide or else be at infinity; $\therefore (ABCD) = -1$ and the range is harmonic. Thus the criterion for a harmonic range ACBD is that the pairs of points C, D and A, B may be *separately* interchangeable, or

$$(ABCD) = (ABDC), \text{ or} = (BACD).$$

As an example of the use of the condition just established we will take the following proposition :—

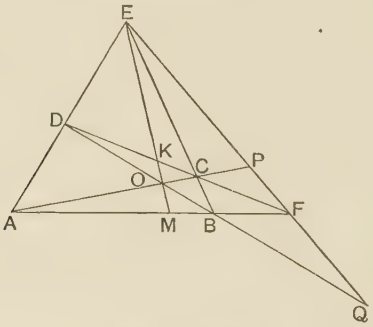
In a complete quadrilateral any diagonal is cut harmonically by the other two diagonals (see fig., p. 150).

Let A, B, C, D, E, F be the vertices of the quadrilateral, and OPQ its diagonal triangle. Join EO, meeting CD, BA in K and M respectively. Then

$$\begin{aligned}(ABMF) &= O(ABMF)\\ &= (CDKF), \text{ since CDKF is another transversal of the same pencil,}\\ &= E(CDKF)\\ &= E(BAMF) = (BAMF);\end{aligned}$$

that is, the points A, B can be interchanged without

altering the value of the cross ratio (ABMF); therefore the range AMBF and all the pencils mentioned above are harmonic, and AOCP, DOBQ, and PEQF are ranges on one or other of these pencils.



INVOLUTION

DEFS.—If a system of pairs of collinear points A, A′; B, B′; C, C′, etc., be so situated with regard to another point O in the same straight line that $OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \text{etc.}$, they are said to be in INVOLUTION. The point O is called the CENTRE, and A, A′; B, B′; C, C′, etc., are called CONJUGATE POINTS of the involution.

The points E, F situated on the range on opposite sides of O such that

$$OE^2 = OF^2 = OA \cdot OA' = OB \cdot OB' = \text{etc.}$$

are called the DOUBLE POINTS of the involution.

If lines be drawn from a point S outside the range to A, A′; B, B′, etc., they form a PENCIL IN INVOLUTION, and SE, SF are called the DOUBLE RAYS of the pencil.

NOTE.—It should be observed that the double points and double rays are real only when the conjugate points of the involution are on the same side of the centre.

Theorem 9

The two double points and any pair of conjugate points of any involution form a harmonic range.

For O is the middle point of EF, and

$$OE^2 = OF^2 = OA \,.\, OA' ;$$

$\therefore$ EAFA′ is a harmonic range (Chap. III. 3).

Theorem 10

In a system of points or rays in involution the cross ratio of any four points or rays is equal to that of their conjugates (*see fig., next page*).

Let $OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = OD \,.\, OD' = OF^2$.

Describe a circle with centre O and radius OF, and take any point S on its circumference for vertex of a pencil through the points in involution.

Since $OA \,.\, OA' = OS^2$, the triangles SOA, OSA′ are similar ;

$\therefore$ angle ASO = SA′O.

Similarly BSO = SB′O ;

$\therefore$ angle ASB = A′SB′.

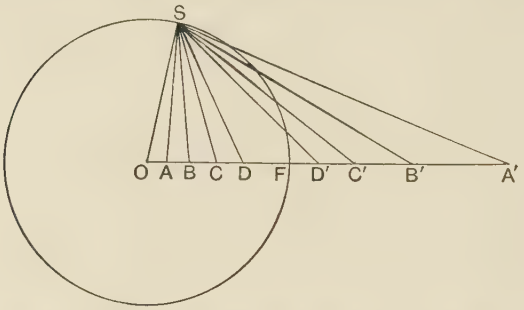
Similarly BSC = B′SC′,

and CSD = C′SD′ ;

$\therefore$ the pencils S(ABCD), S(A′B′C′D′) are equiangular ;

$\therefore$ they are equi-cross (Theorem 4).

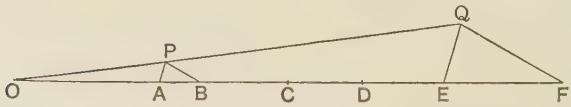
In the case in which the double points are imaginary, and conjugate points on opposite sides of O, a similar



construction can be made by taking S on a circle, the square on whose radius is equal to $-OA \cdot OA'$, etc.

The converse of Theorem 10 may be stated thus:— *If there be six collinear points such that the cross ratio of four of them is equal to the cross ratio of another four, two points only being repeated in these groups of four, and occupying the same relation to each other in both groups, either conjugate or not conjugate, then the six points are in involution.*

Let ABCDEF be the points such that (ACBD) =



(FDEC). Draw AP and EQ parallel, and BP and FQ parallel, and let QP meet the line of points in O. Then

$$OA : OE = OP : OQ = OB : OF ;$$

$$\therefore \text{OA} \cdot \text{OF} = \text{OE} \cdot \text{OB} = k^2 \text{ say};$$

also
$$\frac{\text{AB} \cdot \text{CD}}{\text{CB} \cdot \text{AD}} = \frac{\text{FE} \cdot \text{DC}}{\text{DE} \cdot \text{FC}},$$

or
$$\frac{\text{OB} - \text{OA}}{(\text{OB} - \text{OC})(\text{OD} - \text{OA})} = \frac{\text{OF} - \text{OE}}{(\text{OE} - \text{OD})(\text{OC} - \text{OF})};$$

$$\therefore \frac{k^2(\text{OF} - \text{OE})}{\text{OE} \cdot \text{OF}(\text{OB} - \text{OC})(\text{OD} - \text{OA})} = \frac{\text{OF} - \text{OE}}{(\text{OE} - \text{OD})(\text{OC} - \text{OF})};$$

$$\therefore k^2(\text{OC} \cdot \text{OE} - \text{OC} \cdot \text{OD} - \text{OF} \cdot \text{OE} + \text{OF} \cdot \text{OD})$$
$$= \text{OE} \cdot \text{OF}(\text{OD} \cdot \text{OB} - \text{OA} \cdot \text{OB} - \text{OC} \cdot \text{OD} + \text{OC} \cdot \text{OA});$$
$$\therefore \text{OE} \cdot \text{OF}(k^2 - \text{OC} \cdot \text{OD}) + k^2 \cdot \text{OC} \cdot \text{OD}$$
$$- \text{OE} \cdot \text{OF} \cdot \text{OA} \cdot \text{OB} = 0;$$
$$\therefore \text{OE} \cdot \text{OF}(k^2 - \text{OC} \cdot \text{OD}) + k^2 \cdot (\text{OC} \cdot \text{OD} - k^2) = 0;$$
$$\therefore (k^2 - \text{OC} \cdot \text{OD})(\text{OE} \cdot \text{OF} - k^2) = 0,$$

but k^2 cannot $= \text{OE} \cdot \text{OF}$ unless E and A or F and B coincide, $\therefore k^2 = \text{OC} \cdot \text{OD}$;

$\therefore$ C and D are conjugate points of the involution determined by the pairs of conjugate points A, F and B, E.

As an example of the use of this converse we take the following:—

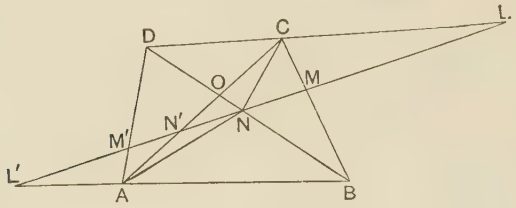
The sides of a complete quadrangle cut any transversal in six points in involution.

Let a transversal cut the sides of the quadrangle ABCD in the points L, M, N, N′, M′, L′. Let the sides AC, BD intersect in O. Join AN and CN. Then

$$\begin{aligned} C(LMNN') &= C(DBNO) \\ &= A(DBNO) \\ &= A(M'L'NN') \\ &= A(L'M'N'N); \end{aligned}$$

$\therefore (LMNN') = (L'M'N'N)$;

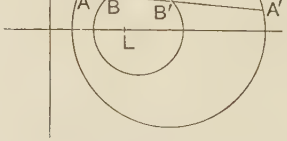
$\therefore$ the six points are in involution.



Theorem 11

A straight line is cut in involution by any system of co-axal circles, the point of intersection of the radical axis with the straight line being the centre of involution.

Let OABB'A' be a transversal cutting a pair of circles in A, A' and B, B', and their radical axis in O. Let



OP, OQ be tangents to the circles from O, and let L be a limiting point of the system. Then

$$OP = OQ = OL;$$

but $OP^2 = OA \cdot OA'$, and $OQ^2 = OB \cdot OB'$;

$$\therefore OA \cdot OA' = OB \cdot OB' = OL^2 = \text{constant};$$

$\therefore$ A, A′, B, B′ are points in involution with O for centre.

If the circles form an intersecting system, with common points X and Y, then we have

$OA \cdot OA' = OB \cdot OB' = \text{etc.} = OX \cdot OY$, which is constant;

and again we get a system of points in involution, but in this case the double points of the system are imaginary.

Theorem A

If a series of pairs of conjugate points with respect to a circle lie on a straight line, then they are in involution, the points where the line cuts the circle being the double points; and if a series of pairs of conjugate lines with respect to a circle are concurrent, they form a pencil in involution, the two tangents being the double rays.

This is evident from the harmonic property of pole and polar (see Chap V. Theorems 6 and C).

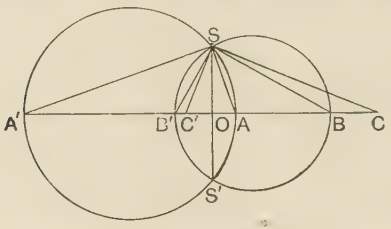
Theorem 12

If two pairs of conjugate rays of a pencil in involution be at right angles, then every pair of conjugate rays are at right angles.

Let SA, SA′; SB, SB′ be pairs of conjugate rays at

right angles. Let a transversal cut the rays in A′, B′, A, B.

On AA′ and BB′ as diameters describe circles, which



will pass through S, since ASA′, BSB′ are right angles, and cut again in S′.

Then SS′ is the radical axis of the circles, and if it intersects the transversal in O, then O is the centre of involution, and $OA \cdot OA' = OB \cdot OB' = -OS^2$.

Again, if the transversal cut any other pair of conjugate rays of the pencil in C and C′,

$$OC \cdot OC' = -OS^2, \quad \text{or } OC : OS = OS : C'O,$$

and the angles at O are right angles;

∴ the triangles COS, SOC′ are similar, and angle CSO = SC′O = complement of OSC′; ∴ SC, SC′ are at right angles.

Cor.—The rays of a series of right angles at the same vertex form a system in involution; for if the rays are cut by any transversal, any pair of rays at right angles intercept on the transversal the diameter of a circle passing through the vertex; and the series of circles so determined form a co-axal system, having

their centres in the transversal, and all passing through the vertex of the pencil.

RECIPROCAL POLARS

DEF.—If with respect to a given fixed circle (centre S, radius k) be taken the $\left\{\begin{matrix}\text{poles}\\\text{polars}\end{matrix}\right\}$ of the $\left\{\begin{matrix}\text{lines}\\\text{points}\end{matrix}\right\}$ of a figure Σ, we obtain a second figure Σ', interchangeably related to the first. Either of the figures Σ, Σ' is termed the POLAR RECIPROCAL of the other; and any geometrical property of the one has its correlative for the other. Thus to the line joining two points of the one corresponds the point of intersection of two lines of the other (Chap. V. Theorem 7); to a number of collinear points, a number of concurrent straight lines; to parallel lines, points collinear with S; to the angle between two lines, the angle (or its supplement) subtended by two points at S, etc.

The given fixed circle is sometimes called the AUXILIARY CIRCLE.

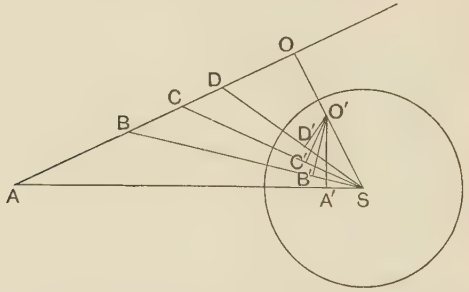
Theorem 13

A triangle may be its own reciprocal polar (Chap. V. Theorem 9).

Theorem 14

A range reciprocates into a pencil of the same cross ratio: and conversely.

If ABCD be the range, SO at right angles to it, and O′ its pole with respect to the auxiliary circle, then the polars of A, B, C, D pass through C′, and the pencils



O′(A′B′C′D′) and S(ABCD) are equiangular, and therefore equi-cross. The converse may be proved in like manner.

Def.—If a line moves in such a way that it is always a tangent to a fixed curve, then the curve is said to be THE ENVELOPE of the moving line.

Theorem 15

If Σ *be a curve, then* Σ′ *is also a curve. To a point on* Σ *corresponds a tangent to* Σ′; *and to a tangent to* Σ *corresponds a point on* Σ′.

If Σ be regarded as the locus of a moving point, then the polar reciprocal Σ′ will be the envelope of a moving line, *i.e.* a curve; and corresponding to any particular position of the point which traces out Σ will be a

particular position of the line of which Σ' is the envelope, *i.e.* a tangent to Σ'.

Again, let P, Q be two points on Σ, and let the corresponding tangents to Σ' intersect in R, then R is the pole with regard to the auxiliary circle of the line PQ (Chap. V. Theorem 7, Cor.), or R in the figure Σ' corresponds to PQ in the figure Σ.

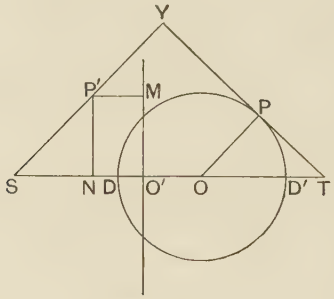
Now let Q approach P until P and Q coincide, so that PQ becomes the tangent to the curve Σ at the point P, then the two corresponding tangents to Σ' will ultimately coincide, and their point of intersection R will be on the curve Σ'. Therefore to a tangent to the curve Σ corresponds a point on the curve Σ'.

Theorem 16

If Σ *be a circle* (*centre* O, *radius* r), *then* Σ' *is a conic, having* S *for its focus,* SO/r *for eccentricity,* k^2/r *for semi-latus rectum, and its transverse axis in the direction* SO. *To the centre* O *corresponds the* S*-directrix; to the tangents at the extremities of the diameter through* S, *the two vertices; to the tangents at* E, E′ *passing through* S, *the points* ω, ω' *at infinity* (Sω, Sω' *being perpendicular to* SE, SE′ *respectively*); *to the points* E, E′ *the asymptotes; to the chord* EE′, *the centre; to the line* FF′ *bisecting* SE, SE′, *the other focus* S′.

Take any point P on the circle and draw TPY the tangent to the circle at P, and SY perpendicular to it. Take P′ in SY, so that $SP' \cdot SY = k^2$, then P′ is the pole of TPY with respect to the auxiliary circle, and the

locus of P′ is the polar reciprocal required. Join SO meeting the circle in D and D′, and PT in T. In SO



take a point O′ so that $SO \cdot SO' = k^2$. Draw P′N, O′M at right angles to SO′, and P′M at right angles to O′M.

Then
$$SP' = \frac{k^2}{SY} = \frac{k^2 \cdot OT}{ST \cdot OP'}$$

and
$$P'M = SO' - SN = SO' - \frac{SP' \cdot SY}{ST} = \frac{k^2}{SO} - \frac{k^2}{ST} = \frac{k^2 \cdot OT}{SO \cdot ST};$$
$$\therefore SP' : P'M = SO : OP,$$

and O′M is a fixed straight line, therefore the locus of P′ is a conic, having S for focus, SO/r for eccentricity, and the polar reciprocal of O for directrix. The conic is an ellipse, parabola, or hyperbola according as $SO < = \text{ or } > r$; that is, according as S is within, upon, or without the circle.

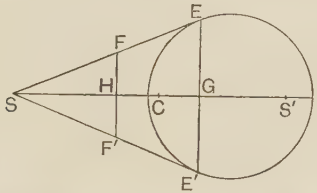
The semi-latus rectum $= SO' \times$ eccentricity
$$= k^2/r.$$

SO is the direction of the transverse axis of the

conic, and its vertices are the reciprocals of the tangents to the circle at D, D′, the extremities of the diameter through S.

Since SE, SE′ pass through the centre S of the auxiliary circle their poles with regard to it will be at infinity in directions Sω, Sω′ at right angles to SE, SE′ respectively. The polar reciprocals of the points E, E′ will be the tangents to the conic at the points at infinity ω, ω′; that is, the asymptotes of the conic. Again, since the asymptotes cut in the centre of the conic, the centre C of the conic must be the polar reciprocal of the chord EE′ of the circle;

∴ if EE′ cut SO in G, $SC \cdot SG = k^2$. Also, if FHF′



be the line bisecting SE, SE′ and cutting SO in H, the polar reciprocal of FF′ will be a point S′ in SO, such that

$$SS' = \frac{k^2}{SH} = \frac{2k^2}{SG} = 2SC;$$

∴ S′ is the other focus of the conic.

Cor.—Since the asymptotes are at right angles to SE, SE′, the angle between them is the supplement of the angle between SE, SE′, the tangents from S to the circle.

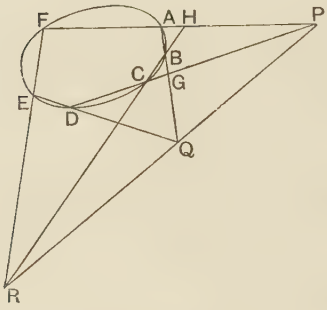
Theorem 17 (*Pascal's Theorem*)

The three points of intersection of the opposite sides of a hexagon inscribed in a conic are collinear.

Let ABCDEF be the hexagon; FA, DC meeting in P; AB, ED in Q; BC, FE in R. Let AB, DC meet in G, and FA, CB in H. Then

$$\begin{aligned} P(ABCQ) &= (ABGQ) \\ &= D(ABCE) = F(ABCE) \text{ (Theorem 7)} \\ &= (HBCR) = P(ABCR); \end{aligned}$$

that is, PQ, PR are in the same straight line.



NOTE.—PQR may be called a Pascal line, and other Pascal lines may be determined by joining the vertices of the hexagon in different orders. To find the number of such lines we may suppose the letters ABCDEF to be permuted, and the same pairs of letters to be joined as in the figure. Keeping one letter at the same point, the others may be permuted in 120 different ways, which occur in pairs of opposite cyclic order, and each pair would give the same Pascal line. There are consequently 60 Pascal lines.

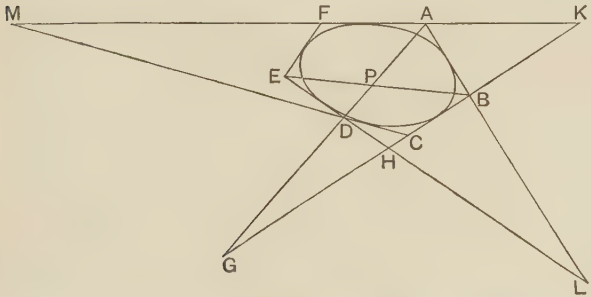
Again, the six vertices have 15 connectors; and 15 straight lines intersect in 105 points; but there are 10

of these points at each of the six vertices, so we see that the 15 connectors intersect in 45 points other than the original six. The theorem is thus more completely enunciated as follows:—The fifteen joins of six points on a conic intersect in forty-five points which lie three by three on sixty straight lines.

Theorem 18 (*Brianchon's Theorem*)

The three joins of the opposite vertices of a hexagon circumscribing a conic are concurrent.

Let ABCDEF be the hexagon, the diagonals AD, BE meeting in P. Let BC meet AD, ED, FA in G, H,



K respectively; let AB, ED meet in L, and AF, CD in M.

Then $(APDG) = B(APDG) = (LEDH)$
$= (AFMK)$ (Theorem 8)
$= C(AFDG)$;

$\therefore$ the ray CF passes through P.

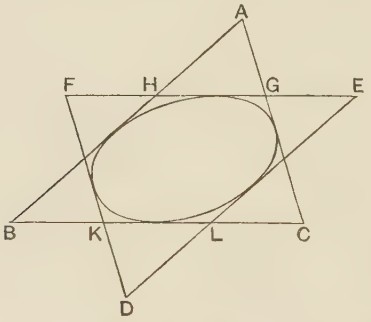
NOTE.—It will be observed that Pascal's and Brian-

chon's theorems are reciprocal polar the one of the other; and the more complete enunciation of the latter is:—The fifteen points of intersection of six tangents to a conic are joined by forty-five straight lines which intersect three by three in sixty points.

Theorem 19

If two triangles circumscribe a conic, their six vertices lie on another conic; and conversely, if the six vertices lie on a conic, the six sides touch another conic.

Let ABC, DEF be the two triangles; and let AC,



AB meet EF in G and H; and DF, DE meet BC in K, L. Then

$$\begin{aligned} A(BCEF) &= (HGEF) \\ &= (BCLK) \text{ (Theorem 8)} \\ &= D(BCEF). \end{aligned}$$

And therefore D is on the conic through A, B, C, E, F (Theorem 7, Cor.)

Again, if A, B, C, D, E, F lie on a conic, then

$$\begin{aligned}(BCLK) &= D(BCEF)\\ &= A(BCEF) \text{ (Theorem 7)}\\ &= (HGEF),\end{aligned}$$

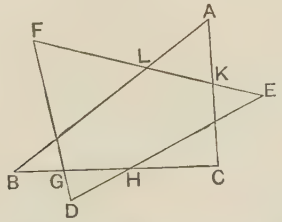
which proves the converse (Theorem 8, Cor.)

Cor.—If two conics are such that a triangle can be drawn so as to be inscribed in one and circumscribed about the other, then an infinite number of such triangles can be drawn.

Let ABC be the triangle circumscribing one conic and inscribed in another. Draw EF a tangent to the former and meeting the latter conic in E, F, and draw the other tangents from E and F meeting in D; then by the proposition D is on the conic through A, B, C, E, F.

Theorem 20

If two triangles are each self-conjugate with respect to a circle, then their sides are tangents to a conic, and their vertices lie on a conic.



Let ABC and DEF be each self-conjugate to a circle. Let BC meet DF, DE in G, H, and EF meet AC, AB in

K, L. Then, since the polars of B, G, H, C form a pencil with the same cross ratio (Theorem 14),

$$(BGHC) = A(CEFB) = (KEFL) = (LFEK).$$

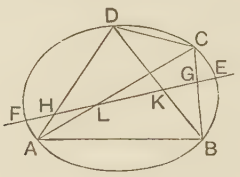
And consequently EF touches the conic which touches BC, CA, AB, FD, DE (Theorem 8, Cor.); and therefore A, B, C, D, E, F lie on a conic (Theorem 19).

NOTE.—It will be seen in the next chapter that as pole and polar properties are projective, this theorem is also true when the triangles are self-conjugate with respect to a conic.

Theorem 21

A conic and the three pairs of opposite sides of any complete quadrangle inscribed in it meet every transversal in four pairs of points in involution.

Let ABCD be the quadrangle, and let any transversal



meet the conic in E, F, BC in G, AD in H, BD in K, AC in L. Then

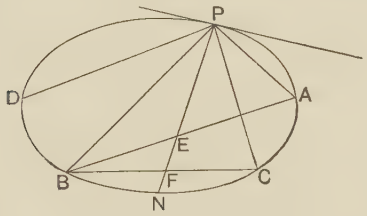
$$(EGKF) = B(ECDF) = A(ECDF) \text{ (Theorem 7)} = (ELHF) = (FHLE).$$

And therefore, by the converse of Theorem 10, E, F; G, H; K, L are pairs of points in involution. And similarly for the pair of points where the transversal meets AB and CD.

Theorem 22

The lines joining the extremities of pairs of chords at right angles drawn from any fixed point on a conic intersect on the normal at the fixed point.

Let PA, PB; PC, PD be two pairs of chords at



right angles, PN the normal at P; and let AB, BC meet PN in E, F. Then

$$\begin{aligned}(\mathrm{PEFN}) &= \mathrm{B}(\mathrm{PACN}) = \mathrm{P}(\mathrm{PACN}) \text{ (Theorem 7)} \\ &= \mathrm{P}(\mathrm{NBDP}), \text{ since corresponding rays are at right angles,} \\ &= \mathrm{C}(\mathrm{NBDP}) \text{ (Theorem 7)} \\ &= \mathrm{C}(\mathrm{PDBN}).\end{aligned}$$

And hence, referring this last pencil to the normal PN, we see that CD must pass through E.

Examples on Chapter VIII

1. If (ABCD) = (ABC'D'), show that (ABCC') = (ABDD').

2. When of six points the joins of any four with the other two form equi-cross pencils, then the joins of *every* four with the remaining two form equi-cross pencils.

3. When of six lines the intersections of any four with the remaining two form equi-cross ranges, then the intersections of *every* four with the remaining two form equi-cross ranges.

4. If A, B, C and D, E, F are points lying on two straight lines, then the intersections of AE and BD, of BF and CE, and of AF and CD are collinear.

5. Given three points in each of two ranges (the joins of the corresponding pairs not being concurrent), determine the point in the second range which will give the same cross ratio as is given by a fourth given point on the first range.

6. The side AB of a triangle ABC is bisected in X, AX is bisected in Y, CA is bisected in Y', CY' in X'; prove that XX', YY', BC are concurrent.

7. A variable line passes through a fixed point; show that the locus of a point on the line forming a range of constant cross ratio with the three points where the line cuts the sides of a given triangle is a conic circumscribing the triangle and passing through the fixed point.

8. S is a point on the circumcircle of the triangle ABC; SL, SM, SN perpendicular to SA, SB, SC respectively, meet BC, CA, AB respectively in L, M, N. Prove L, M, N collinear with the circumcentre.

9. ASB and CSD are perpendicular focal chords of a conic; show that the pencil formed by joining A, B, C, D to any point on the curve is harmonic.

10. Given two pairs of conjugate points of a system in involution, find the centre of the system.

11. Given four collinear points, show that they determine three involutions ; also that the double points of any one of these involutions are harmonic conjugates with respect to the double points of any other.

12. If A, A′ ; B, B′ be two pairs of conjugate points in a system in involution, of which E, F are the double points, then AB, A′B′ subtend equal angles at every point on the circle described on EF as diameter.

13. A variable circle passing through a fixed point, and having its centre on a given straight line, cuts the line in a system of points in involution.

14. The straight lines joining any point to the six vertices of a complete quadrilateral form a pencil in involution.

15. Every circle (or conic) and any two pairs of opposite sides of any inscribed quadrilateral determine six points in involution on every straight line.

16. The three sides of any triangle and any three concurrent straight lines through the vertices determine six points in involution on every straight line.

17. The three vertices of any triangle and any three collinear points on the sides determine a pencil in involution at every point.

18. The three intersections with any line of the sides of a triangle determine, with the projections on the line of the vertices, a system of six points in involution.

19. The joins of any point with the vertices of a triangle, and the parallels to the sides through the point, determine a pencil in involution.

20. Every line through their radical centre intersects three circles in six points in involution.

Deduce the following theorems from one another by reciprocation :—

21. The tangent to a circle at a point is at right angles to the radius drawn to that point.

Any point on a conic and the intersection of the tangent at that point with the directrix subtend a right angle at the focus.

22. Tangents to a circle make equal angles with their chord of contact.

Tangents to a conic subtend equal angles at the focus.

23. The polar of a point with respect to a circle is at right angles to the line joining the point to the centre.

Any point and the intersection of its polar with the directrix subtend a right angle at the focus of a conic.

24. Show how to reciprocate co-axal circles into confocal conics.

Show that the following theorems are reciprocal :—

25. The common tangent to two circles subtends a right angle at either limiting point.

Confocal conics cut at right angles.

26. If a line cut two circles, the intercepts between the circles subtend equal angles at either limiting point.

Tangents from a point to two confocal conics are equally inclined to each other.

27. The polar of a fixed point with regard to co-axal circles passes through another fixed point, and the two points subtend a right angle at either limiting point.

The poles of a fixed straight line with regard to confocal conics lie on another fixed straight line at right angles to the given line.

28. Prove that two similar and similarly situated central conics, having a common focus lying between their centres, can be reciprocated into two circles having the same centres as the conics, but interchanged.

29. Reciprocate with respect to O: The envelope of a chord of a fixed circle which subtends a constant angle at a fixed point O of its circumference is a concentric circle.

30. Reciprocate Feuerbach's theorem, first, with respect to the orthocentre, second, with respect to any point.

31. Given five points, show how to draw by ruler only the tangents at these points to the conic passing through them.

32. Given five straight lines, show how to find by ruler only the points where the conic they envelop touches these lines.

33. TP, TQ, PQ are three tangents to a conic, T being situated on another conic having the same focus and directrix; show that the angle subtended at the common focus by PQ is constant.

34. SP and SQ are tangents to a conic; a line through Q cuts the conic in B and SP in A, the tangent at B meets SP in C; show that SACP is a harmonic range.

35. A series of parabolas is described having a common focus and a common tangent, a fixed straight line is drawn parallel to their common tangent cutting the parabolas; prove that the envelope of the tangents at the points of intersection is a conic.

36. Show, by reciprocation, that if the diagonals of a cyclic quadrilateral are at right angles, and intersect in a fixed point, then the sides touch a fixed conic.

CHAPTER IX

PROJECTION

Definitions

If two figures in different planes be so related to a fixed point outside both planes that the straight line joining this fixed point with any point in one figure passes through a point in the other figure, each figure is said to be the CENTRAL or CONICAL PROJECTION of the other.

The fixed point is called the CENTRE or VERTEX of projection, and the intersection of the planes the AXIS of projection.

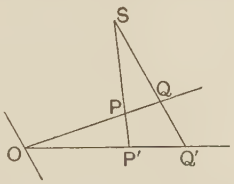
Points in which any straight line through the centre of projection meets the two planes are called CORRESPONDING POINTS.

When the centre of projection is at an infinite distance the projection becomes PARALLEL.

Theorem 1

The projection of a straight line is a straight line which intersects the former in the axis of projection.

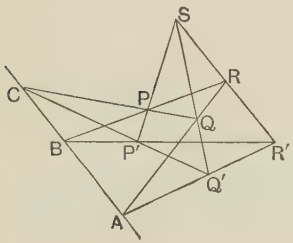
The plane through the centre of projection S and the given straight line OPQ cuts any other plane in a straight line OP′Q′ which meets the axis of projection in the same point O as the given line, and contains the projection P′ of any point P on OPQ; that is, the straight lines are the projections of one another.



Cor.—The straight line joining two points projects into the straight line joining the corresponding points; the intersection of two lines projects into the intersection of the corresponding lines; and the tangent to a curve at any point projects into the tangent to the corresponding curve at the corresponding point.

Theorem 2

If two figures in different planes are so related that corresponding straight lines meet (in the axis), then the straight lines joining corresponding points are concurrent.



Let ABC be the line of intersection of the two planes; let QR, Q′R′ meet in A; RP, R′P′ in B; and PQ, P′Q′ in C. Then the planes AQR′, BRP′, CPQ′ meet in a point S, and this point S is determined by

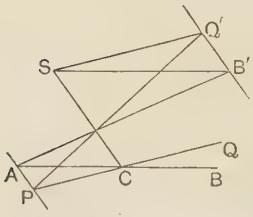
two of the three lines PP′, QQ′, RR′, and is ∴ independent of the position of one of the points A, B, C in the line ABC; ∴ if we regard S as determined by PP′, QQ′, the line RR′ joining any other pair of corresponding points will also pass through S.

DEFS.—If a plane be drawn through the centre of projection parallel to the plane containing one of the figures, its intersection with the other plane is called the VANISHING LINE, and may be considered to be the line corresponding to the line at infinity of the first plane.

The intersection of a line in either figure with the vanishing line in its plane is termed its VANISHING POINT, and corresponds to the point at infinity on the corresponding line in the other figure.

Theorem 3

Any line of one figure is parallel to the line joining the centre with the vanishing point of the corresponding line in the other.



Let PQ be the line of one figure meeting the axis in P, and PQ′ the corresponding line of the other figure meeting the vanishing line in Q′; then SPQQ′ is a plane cutting two parallel planes in the lines SQ′, PQ, ∴ SQ′, PQ are parallel.

Theorem 4

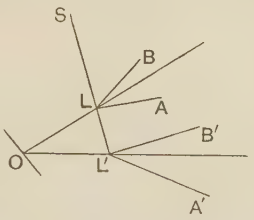
The angle contained by any two straight lines in one figure is equal to the angle subtended at the centre by the vanishing points of the corresponding lines in the other.

In the figure of Theorem 3, let AB, AB′ be another pair of corresponding lines, and let AB cut PQ at C. Then PQ, SQ′ are parallel and AB, SB′ are parallel; ∴ angle BCQ = B′SQ′.

Theorem 5

If through the centre and in the plane perpendicular to the axis of projection a straight line be drawn making equal angles with the planes of the two figures, the two corresponding points thus determined are such that the angle contained by any pair of lines through one is equal to the angle between the corresponding lines.

Let L and L′ be the corresponding points so determined, and O the point in which the plane through S perpendicular to the axis meets the axis. Then the angles OLL′, OL′L are equal; ∴ OL = OL′; ∴ if one plane revolves round the axis until the two coincide, L and L′ will coincide, and corresponding lines through L and L′ will coincide; ∴ pairs of corresponding lines through L and L′ contain equal angles.



COR. and DEF.—Two such points may be found in each plane, and they are called LIMITING POINTS.

Theorem 6

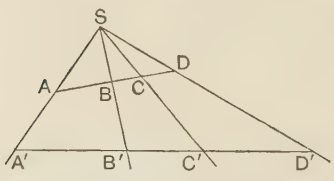
Parallel lines correspond to lines intersecting on the vanishing line.

The point of intersection of parallel lines is at infinity ; ∴ the point of intersection of the corresponding lines is on the vanishing line.

Theorem 7

Any range or pencil of four elements projects into an equi-cross range or pencil ; and any range of points or pencil of rays in involution projects into a range or pencil in involution.

If ABCD and A′B′C′D′ are corresponding ranges,



and AA′, BB′, CC′, DD′ meet in the vertex S, the cross ratio of each range is that of the pencil S(ABCD), or they have the same cross ratio ; since this is also true if ABCD, A′B′C′D′ are transversals to a pair of corresponding pencils in the two planes, ∴ a pencil projects into an equi-cross pencil ; and if a range or pencil of rays in involution be projected, since the cross ratio of any four elements

is unaltered by projection, the projection will also be a range or pencil in involution.

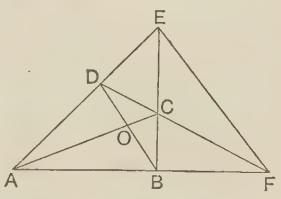
COR.—In particular, a harmonic range or pencil projects into a harmonic range or pencil.

Theorem 8

Any two angles of a figure can be projected into angles of given magnitude, and at the same time a straight line of the figure can be projected to infinity.

Suppose the arms of the two chosen angles to meet the line that is to be projected to infinity in A, B and C, D respectively; take any plane through this line and in it draw segments of circles on AB and CD, containing angles respectively equal to those into which the chosen angles are to be projected, then either point of intersection of the segments may be taken as centre of projection, if the plane on to which the figure is to be projected be taken parallel to the plane containing the segments.

COR.—Any quadrilateral can be projected into a parallelogram of any form and size. Let A, B, C, D, E, F be the vertices of the quadrilateral. Project the diagonal EF to infinity, then the projection will be a parallelogram. Project the angles A and O into the corresponding angles of the



required parallelogram; they will determine the shape of the parallelogram, and the required size can be obtained by taking the plane of projection the necessary distance from the centre.

Theorem 9

A circle can be projected into a conic of any eccentricity and magnitude.

From Chap. VIII. Theorem 4, Cor., if P be a variable point and A, B, C, D fixed points on the circle, the cross ratio P(ABCD) is constant; ∴ by Theorem 7 the projection of the circle is the locus of the point at which four fixed points subtend a pencil of constant cross ratio; that is, by Chap. VIII., Theorem 7, Cor., a conic through the four points.

Considering only the cases where the cone of projection is a right circular cone, we know that the eccentricity of a section may be anything not greater than the eccentricity of the hyperbola, the angle between whose asymptotes is equal to the vertical angle of the cone, and as this angle may be of any magnitude less than two right angles, we see that even in this restricted case the projection may have any eccentricity. The magnitude of the section may be varied indefinitely by taking different parallel sections.

Cor.—The conic will be a hyperbola, parabola, or ellipse according as the vanishing line cuts, touches, or does not meet the circle. For if the vanishing line cuts

the circle in A and B, the projection has two corresponding points A′ and B′ at infinity, and real tangents at infinity parallel to SA, SB; that is, the curve is a hyperbola. If the vanishing line touches the circle, the projection has a single point at infinity where it is touched by the line at infinity corresponding to the vanishing line; that is, the curve is a parabola. If the vanishing line does not meet the circle, the projection has no points at infinity; that is, the curve is an ellipse.

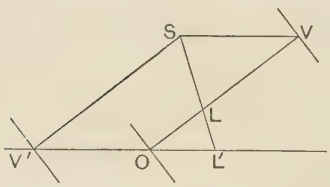
NOTE.—The definition of Pole and Polar and the properties proved in Theorems 6-8, 10-12 of Chap. V., and Theorem A, Chap. VIII., obtain for any conic, being all projective. Also the principle of reciprocal polars may be extended by taking any given fixed conic instead of a fixed circle; and the $\left\{\begin{matrix}\text{poles}\\\text{polars}\end{matrix}\right\}$ of the $\left\{\begin{matrix}\text{lines}\\\text{points}\end{matrix}\right\}$ of a figure Σ with respect to this conic give a second figure Σ', interchangeably related to the former. Either of the figures Σ, Σ' is termed the polar reciprocal of the other with respect to this conic. Theorems 14 and 15, Chap. VIII., are still true, and it is easily seen from them and from Theorems 7 and 8, Chap. VIII., that the reciprocal polar of any conic Σ with respect to a fixed conic S is a conic Σ'. Thus is the principle of duality completed with regard to conics.

Theorem 10

When the plane of one figure is turned round the axis of

projection, the figures are still projections of one another, and the locus of the centre of projection is a circle; and when the two planes coincide, the centre of projection coincides with one or other of the limiting points.

The figures are still projections of one another, for corresponding straight lines will still meet in the axis, and therefore the lines joining corresponding points will still be concurrent.



Again, the plane through S perpendicular to the axis cuts the vanishing lines in V, V′ and the axis in O, forming a parallelogram SVOV′.

Let OV′ revolve about the axis, then SVOV′ will still be a parallelogram; and since OV and OV′ are of fixed length, S will describe a circle about V as centre; also, if L, L′ are the limiting points, the

$$\text{angle } VSL = OL'L = OLL' = VLS;$$

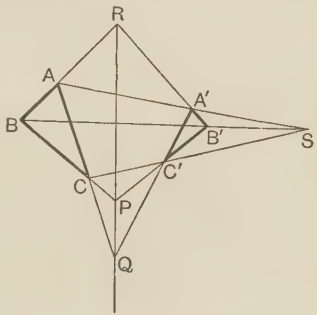
.. VS = VL, and when the planes coincide S will coincide with L.

DEFS.—When the two planes thus coincide, the figures are said to be in PLANE PERSPECTIVE or HOMOLOGY, and the axis and centre are called the AXIS and CENTRE OF PERSPECTIVE, or HOMOLOGY.

Theorem 11

If the lines joining corresponding vertices of two triangles are concurrent, then the intersections of corresponding sides are collinear; and conversely, if the intersections of corresponding sides of two triangles are collinear, then the lines joining the corresponding vertices are concurrent.

If ABC and A′B′C′ be different planes, intersecting in the straight line PQR, and AA′, BB′, CC′ meet in S, then the pairs of corresponding lines BC, B′C′; CA, C′A′; AB, A′B′ will meet in the line PQR, and this will still be true if the planes revolve until they are in the same plane with PQRS; the converse may be proved in like manner by the aid of Theorem 2.



NOTE.—This theorem is commonly stated thus:—*Co-polar triangles are also co-axial; and, conversely, co-axial triangles are co-polar.* Another proof is subjoined.

Since B′A′R is a transversal to the triangle ABS,

$$BB' \cdot SA' \cdot AR = -B'S \cdot A'A \cdot RB;$$

since B′C′P is a transversal to the triangle BCS,

$$B'S \cdot C'C \cdot PB = -SC' \cdot CP \cdot BB';$$

and since A′C′Q is a transversal to the triangle ACS,

$$A'A \cdot QC \cdot C'S = -AQ \cdot CC' \cdot SA'.$$

Multiplying these results we get

$$AR \cdot BP \cdot CQ = -RB \cdot PC \cdot QA;$$

∴ P, Q, R are collinear.

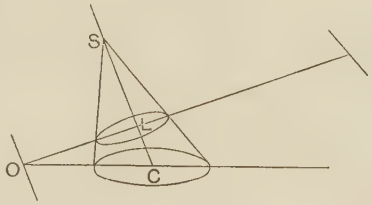
For the converse: let AA′, CC′ meet in S, then we have to prove B, B′, S collinear; this follows from the first case, for the lines joining corresponding vertices of the triangles RAA′, PCC′ are concurrent in Q, therefore B, B′, S, the intersections of corresponding sides, are collinear.

Theorem 12

The perspective of a circle whose centre is the centre of perspective is a conic whose focus is the centre of perspective, and whose directrix is the vanishing line of its own figure.

Let C be the centre of the circle, and CO perpendicular to the line of intersection of the plane of the circle and any other plane; in this plane draw OL at right angles to the line of intersection and equal to OC. Join CL, then if we take any point S on CL for centre of projection, the projection of the circle is a conic, and C, L are the limiting points in the planes, so that any conjugate lines of the circle through C project into conjugate lines of the conic at right angles and passing through L; ∴ L is a focus of the conic, the foci being the only points through which conjugate lines are at

right angles, and its polar is the directrix, but its polar is the projection of the polar of C with respect to the circle, that is, of the line at infinity; ∴ the directrix is the vanishing line of its own figure. Now let the plane of the circle revolve about the line of intersection until the planes coincide, then C, L, and S coincide, and we obtain the required result.



NOTE ON IMAGINARY POINTS AND LINES

As a straight line which cuts a circle or conic does so in two and only two points, it is usual and convenient in modern geometry to say that EVERY straight line in the plane of a circle or conic cuts it in two points real, coincident, or imaginary; a tangent cutting it in two coincident points, and a line which does not meet it being said to cut it in two imaginary points. The second point where a diameter of a parabola, or a line parallel to an asymptote of a hyperbola cuts the curve is at infinity, and the asymptotes of a hyperbola meet the curve at two points at infinity; and we might thus speak of the imaginary asymptotes of an ellipse. Many theorems in this book may be generalised by introducing the idea of imaginary points and lines:

thus (see Chap. V. Theorem 8, Cor.) the polar of a point may always be regarded as the chord of contact of tangents real, coincident, or imaginary drawn from that point. Again (see Chap. VI. Theorem 6, Cor.), the radical axis of two circles is the common chord of the circles whether they meet in real or imaginary points. Or again, when the conjugate points or rays of a system in involution are on opposite sides of the centre, the double points or rays become imaginary. Further, if the conception be allowed that imaginary points and lines may be projected into real points and lines, the value and power of the principle of projection becomes greatly extended.

The following particulars should be noted:—

That imaginary points occur in pairs, so that to every imaginary point corresponds another imaginary point which might be called its conjugate. The straight line joining a pair of conjugate imaginary points is real, and no other straight lines through either imaginary point can be real.

Two conics in the same plane intersect in four points of which all may be real, two real and two imaginary, or all imaginary; in the first case the six chords of intersection are all real, and in the latter two cases two only of the chords are real, the other four being imaginary.

In the case of two intersecting circles, the second real chord of intersection is the line at infinity (see Theorem 13 below). When two circles in a plane do not intersect, the two real chords of contact are the

radical axis and the line at infinity, and these coincide in the case of concentric circles (see Theorem 15 below).

Theorem 13

Every circle in a plane passes through the same two imaginary points on the line at infinity.

Let P, Q be two points at infinity in directions at right angles, then P, Q are conjugate points with regard to any circle in the plane. Let a circle with centre O cut the line at infinity in the imaginary points ϕ, ϕ'; then ϕ, ϕ' must be harmonic conjugates of P, Q (Chap. V. Theorem 6). Similarly, if R, S be any other pair of points at infinity in directions at right angles, ϕ, ϕ' are harmonic conjugates of R, S. Hence ϕ, ϕ' are the foci of the involution which the arms of any right angle revolving in the plane determine upon the line at infinity; that is, ϕ, ϕ' are fixed points, and every circle in the plane passes through them.

Cor.—Every right angle POQ is divided harmonically by the lines $O\phi$, $O\phi'$.

Def.—These two points in any plane are called the CIRCULAR POINTS at infinity, or the FOCOIDS. (The latter term is due to Dr. C. Taylor. See Taylor's *Ancient and Modern Geometry of Conics*, p. 308.)

Theorem 14

Every conic which passes through the focoids is a circle.

If C be the centre of a conic, and ϕ ϕ' the focoids, then $\phi\phi'$ is the polar of C or chord of contact of tangents from C to the conic. But if CP, CQ be any two lines at right angles, P, Q are harmonic conjugates of ϕ, ϕ', and $\therefore$ CP, CQ are conjugate diameters of the conic, and the only conic whose conjugate diameters are at right angles is the circle.

Theorem 15

Concentric circles touch each other at the focoids.

If C be the centre and ϕ, ϕ' the focoids, then $\phi\phi'$ is the polar of C or chord of contact of tangents from C to any circle with centre C. Hence Cϕ, Cϕ' are tangents to every circle whose centre is C, or the circles touch each other at the focoids ϕ, ϕ'.

Theorem 16

Co-axal circles may be projected into a system of conics passing through four fixed points; and concentric circles may be projected into a system of conics touching one another at two fixed points.

Co-axal circles all pass through four fixed points, namely, the two points in which they cut their radical axis and the two focoids; hence the projections of the circles will be conics passing through four fixed points.

Concentric circles all touch at the focoids, hence the projections will be conics touching at two fixed points —the projections of the focoids.

Theorem 17

Every conic may be considered as inscribed in a quadrilateral which has for its six vertices the two real foci, the two imaginary foci, and the two focoids.

If S, S′ be the foci, and ϕ, ϕ' the focoids, then, since all pairs of lines through S conjugate with respect to the conic are at right angles, they form a pencil in involution, of which $S\phi$, $S\phi'$ are the double rays, and these latter are consequently tangents to the conic (Chap. VIII. Theorem A). And similarly for $S'\phi$, $S'\phi'$. The two other vertices Σ, Σ' of the quadrilateral thus formed are necessarily imaginary, and are called the imaginary foci. The sides of the diagonal triangle of the quadrilateral are SS′, $\Sigma\Sigma'$, and the line at infinity, and they are all real lines. The intersection of SS′, $\Sigma\Sigma'$ being the pole of the line at infinity is the centre of the conic, and $\Sigma\Sigma'$ being a conjugate line to the transverse axis SS′, must be the conjugate axis.

Examples on Chapter IX

1. In what circumstances is the projection of a circle another circle?

2. Show how to project a triangle of any shape into an equilateral triangle.

3. A conic cuts the sides BC, CA, AB of a triangle ABC in a, a'; b, b'; c, c' respectively. Show that

$$Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca' = Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb'.$$

(Carnot.) [Project the conic into a circle.]

4. If four points on a conic subtend a harmonic pencil at any fifth point on the conic, the point of intersection of the tangents at each pair of conjugate points lies on the chord joining the other pair. [Project into a circle.]

5. If four tangents to a conic cut any fifth tangent harmonically, the chord of contact of each pair of conjugate tangents passes through the point of intersection of the other pair. [Project into a circle.]

6. When the sides of one figure are respectively parallel to those of another figure, the lines joining corresponding vertices are concurrent.

7. If two circles are in perspective, where is the axis of perspective?

8. When three triangles are in perspective, two and two, and have a common axis of perspective, their three centres of perspective are collinear.

9. When three triangles are in perspective, two and two, and have a common centre of perspective, their three axes of perspective are concurrent.

10. Every two triangles reciprocal polar to each other with respect to a circle (or conic) are in perspective.

11. ABCD is a quadrilateral inscribed in a conic; show that the intersections of CA and DB, of CB and AD, and of the tangents at C and D are collinear.

12. ABC is a triangle inscribed in a conic; show that the tangents at A, B, C meet the opposite sides in collinear points.

13. A system of conics inscribed in a quadrilateral can be projected into confocal conics.

14. If two conics touch at two points, any chord of one which touches the other is cut harmonically at the point of contact and the point where it meets the chord of contact of the conics. [Project into concentric circles.]

15. If each of a system of conics touches two fixed straight lines at fixed points A, B, the poles with regard to the conics of any given straight line will lie on a second straight line, and these two straight lines will divide AB harmonically. [Project into concentric circles.]

16. Conics through four fixed points cut any transversal in pairs of points in involution. [Project into co-axal circles.]

17. The polars of any point with respect to four conics which pass through four given points form a pencil of constant cross ratio. [Project into co-axal circles.]

18. Two given straight lines OAB, OCD, drawn parallel to a pair of conjugate diameters of a conic, meet it in A, B, C, D, and P is any other point on the conic; any transversal meets the pencil P(ABCD) in the points a, b, c, d. Prove that the cross ratio $\frac{ac \,.\, bd}{bc \,.\, ad}$ is equal to $\frac{OB \,.\, OC + OA \,.\, OD}{OA \,.\, OC + OB \,.\, OD}$.

[Prove for a circle and project.]

19. Project the theorem that if two circles cut in real points, the common chord bisects the common tangents.

20. The reciprocal of a conic is another conic, hyperbola, parabola, or ellipse according as real, coincident, or imaginary tangents can be drawn to the given conic from the centre of reciprocation.

21. The reciprocal of a parabola with regard to a point on the directrix is a rectangular hyperbola.

22. Show by reciprocating Theorem 11, Chap. VI., that the locus of the centres of conics inscribed in a quadrilateral is a straight line.

23. Reciprocate with regard to the orthocentre the theorem that all rectangular hyperbolas circumscribing a triangle pass through the orthocentre.

24. A conic touches the sides BC, CA, AB of a triangle ABC in D, E, F respectively; D′, E′, F′ are the other extremities of the diameters through D, E, F; show that AD′, BE′, CF′ are concurrent.

25. In each of a series of conics through four points a diameter is drawn conjugate to a given straight line. Prove that these diameters are concurrent.

26. ABC is a triangle; AD, BE, CF are drawn to BC, CA, AB so that AD, BC; BE, CA; CF, AB are parallel to conjugate diameters of any conic in the plane of the triangle; prove AD, BE, CF concurrent.

27. Two conics have double contact, O being the point of intersection of common tangents; from points P and Q on the outer conic pairs of tangents are drawn to the inner forming a quadrilateral; and R is the pole of PQ with regard to the inner conic. Prove that two diagonals of the quadrilateral pass through R, and that one of these diagonals passes through O.

28. From a point T two tangents TP, TP′ are drawn to a conic, and TQ, TQ′ to another conic which touches the first at fixed points A and B; show that PQ, PQ′, P′Q, P′Q′ all touch another conic which also touches the two former conics at A and B.

MISCELLANEOUS EXAMPLES

1. Place two given rectilinear segments in the same straight line so that one shall cut the other harmonically.

2. On a given straight line find a point such that the given straight line and the straight lines drawn from the point to the vertices of a given triangle shall form a harmonic pencil.

3. Prove, by inversion, that if a triangle ABC has a right angle at C, and CD be perpendicular to AB, then

$$\frac{1}{CD^2} = \frac{1}{CA^2} + \frac{1}{CB^2}.$$

4. Invert the theorem that one circle can be described about a square and another circle inscribed in the square, the centres of the circles being the intersection of the diagonals of the square.

5. In a system of non-intersecting co-axal circles, find that circle which cuts a given fixed circle orthogonally.

6. Find a point on the hypotenuse of a right-angled triangle such that the sum of the squares of its distances from the sides may be a minimum.

7. B, B′ ; C, C′ ; D, D′ ; E, E′ are four pairs of harmonic conjugates with respect to the same pair of points A, A′ ; prove that

$$(BCDE) = (B'C'D'E').$$

8. If A, B are harmonic conjugates with respect to C, D, and P is any other point on the same range, then

$$(ABCP) + (ABDP) = 0.$$

9. Prove that the midpoints of the six joins of four points lie on a conic.

10. Given the centre of a conic and three points on it, construct the tangent to the conic at one of the points.

11. A quadrilateral ABCD is inscribed in a parabola; through two of its vertices C and D straight lines are drawn parallel to the axis meeting DA, BC in P and Q; show that PQ is parallel to AB.

12. Given five points, find the points in which the conic through them would cut any given straight line.

13. ABC is a triangle and L, M, N the feet of the altitudes. Three conics are described, one touching BM, CN at M, N and passing through A; a second touching CN, AL at N, L and passing through B; and a third touching AL, BM at L, M and passing through C; prove that at A, B, C they all touch the same conic.

14. Find the direction of the axis of a parabola which touches four given straight lines.

15. Through any point O in the plane of a triangle ABC are drawn OA′, OB′, OC′ bisecting the supplements of the angles BOC, COA, AOB respectively, and meeting BC, CA, AB in A′, B′, C′ respectively; show that A′, B′, C′ are collinear, and that OA, OB, OC, OA′, OB′, OC′ form a pencil in involution.

16. A, B, C are three points on a straight line; A′ is the harmonic conjugate of A with respect to BC, B′ of B with respect to CA, and C′ of C with respect to AB; show that AA′, BB′, CC′ are pairs of conjugate points in involution.

17. Determine the envelope of a straight line which meets the sides of a triangle in P, Q, R so that the ratio PQ : PR is constant.

18. Through a point of intersection of two circles draw a straight line such that the chords intercepted on it by the circles may be in a given ratio ; prove that two such lines can be drawn, and that the pairs of such lines corresponding to different values of the given ratio form a system in involution.

19. Pairs of tangents to a conic which intersect on a given straight line cut any other fixed tangent in a system in involution.

20. When two quadrilaterals are inscribed in a conic, if three sides of the first meet respectively three sides of the second in three collinear points, the point of intersection of the fourth pair is also collinear with them.

21. Two similar and similarly situated conics have a common focus which is not a centre of similitude ; prove that a parabola can be described touching the common chord and common tangents of the conics, and having its focus at their common focus.

22. Reciprocate the proposition that the bisectors of the angles of a triangle intersect three by three at the centres of the circles touching the sides.

23. Generalise by reciprocation :—Tangents to a parabola at right angles to each other intersect on the directrix.

24. Reciprocate the proposition : -The locus of the centre of a circle C which touches two concentric circles A and B is a third concentric circle ; and the points of contact of C with A, B are collinear with the centre of C and the common centre of A, B.

25. Reciprocate with regard to a limiting point of the

two fixed circles the proposition that, if a circle touches two fixed circles, the join of the points of contact passes through a centre of similitude of the fixed circles.

26. P is any point on a conic whose focus is S, and a line through S which makes a given angle with SP meets the tangent at P in T; show that the locus of T is a conic whose focus and directrix coincide with that of the original conic.

27. Reciprocate Theorem 9, Chap. VI.

28. ABCD is a cyclic quadrilateral; if the reciprocal polar of the triangle ABC taken with regard to D be *abc*, prove that *abc*D is cyclic.

29. Reciprocate :—If a line is drawn from the focus of an ellipse making a constant angle with the tangent, the locus of its intersection with the tangent is a circle.

30. Reciprocate :—If two equal circles are traced in a plane, two of their common tangents are parallel.

31. The opposite vertices AA′, BB′, CC′ of a quadrilateral circumscribing a conic are joined to a given point O; OA cuts the polar of A in a, OB cuts the polar of B in b, and so on; show that a conic can be drawn through the seven points O, a, a', b, b', c, c' (*Educational Times*, vol. lix. No. 11,740, p. 25).

32. A tangent to a conic cuts two fixed tangents in T and T′; R, R′ are two fixed points; show that the locus of the intersection of TR and T′R′ is a conic.

33. If AB, CD be two conjugate chords of a conic and P any point on it, P(ABCD) is a harmonic pencil.

34. A series of equal, similar, and concentric ellipses are reciprocated with respect to a circle; show that if one of the reciprocals be a rectangular hyperbola they will all be so, and have double contact with a hyperbola whose eccentricity

e' is given by $e'^2 + e^2 = 2$, e being the eccentricity of the ellipses.

35. Straight lines are drawn through a fixed point on a conic making equal angles with a fixed direction and meeting the curve again in P and Q; show that PQ passes through a fixed point.

36. If a conic circumscribe an equilateral triangle, and have the centre of the triangle as its focus, show that its latus rectum is one-third of the diameter of the circumcircle, and that its eccentricity is $\frac{4}{3}$.

37. A rectangular hyperbola passes through the angular points, and a parabola touches the sides of a given triangle; show that the tangents drawn to the parabola from one of the points where the hyperbola cuts the directrix of the parabola are parallel to the asymptotes of the hyperbola. Which of the two points on the directrix is to be taken? When the two points coincide, show that one curve is the polar reciprocal of the other with regard to the coincident points.

38. With the orthocentre of a triangle as focus, two conics are described touching a side of the triangle and having the other two sides as directrices respectively; prove that their conjugate axes are equal.

39. P is a point on a conic, S the focus, PS meets the corresponding directrix in p, and the tangent at P meets that directrix in p'; show that for different positions of P, p, p' will be conjugate points in a system in involution.

40. The cross ratio of the pencil formed by joining a point on one of two conics to their four points of intersection is equal to the cross ratio of the range on a tangent to the other by their four common tangents.

41. A polygon is inscribed in an ellipse so that each

side subtends the same angle at one of the foci ; prove that if the alternate sides be produced to meet, their points of intersection will lie on a conic having the same focus and directrix as the original ellipse, and that the chords joining the consecutive points of intersection all subtend the same constant angle at the focus as the sides of the original polygon.

42. AB, PQ are a pair of conjugate chords of a conic ; QR is drawn parallel to AB, meeting the conic in R ; show that PR bisects AB.

43. ABC is a triangle circumscribing a conic whose focus is S ; lines are drawn through S perpendicular to SA, SB, SC, meeting another tangent in a, b, c respectively ; show that Aa, Bb, Cc are concurrent.

44. Four chords OAA′, OBB′, OCC′, ODD′ are drawn from any point O to a conic ; show that the two conics through OABCD and OA′B′C′D′ touch each other at O. Show also that a conic can be drawn touching AB, BC, CD, A′B′, B′C′, C′D′.

45. One conic touches OA, OB in A and B, and a second conic touches OB, OC in B and C ; prove that the other common tangents to the two conics intersect on AC.

46. Conics circumscribing a triangle have a common tangent at the vertex ; through this point a straight line is drawn. Show that the tangents at the various points where it cuts the curves all intersect on the base.

47. Show that in general two conics can be found which pass through four given points, and are the polar reciprocals of each other with regard to a certain circle. Also give a geometrical construction for finding the centre of this circle.

48. If conics pass through two fixed points and touch at another fixed point, the common tangents to any

pair of them intersect on a straight line passing through the point where the conics touch each other.

49. Generalise by projection :—If CA, CB are tangents to a parabola, and from any point D in the chord of contact AB, DE, DF are drawn parallel to the tangents and meeting them in E, F, then EF is a tangent to the parabola.

50. If two conics have contact of the third order, a tangent to one is divided harmonically at the point of contact and its points of intersection with the other curve and the common tangent.

51. Generalise the following proposition (1) by reciprocating with respect to A, and (2) by projection :—

A fixed circle whose centre is O touches a given straight line at a point A ; the locus of the centre of a circle which touches the fixed circle and fixed straight line is a parabola whose focus is O and vertex A.

52. Find the locus of the centres of rectangular hyperbolas which pass through three given points.

53. Generalise by projection Euclid III. 32.

54. Project the theorem that one circle can be described about a square and another inscribed in it, the centres of both being the intersection of the diagonals.

55. Project a given conic into a circle so that a focus of the conic is projected into the centre of the circle, and hence prove that (1) the segment intercepted on any tangent by two fixed tangents to a conic subtends a constant angle at the focus, and (2) the envelope of a chord of a conic which subtends a constant angle at a focus is a conic having the same focus and directrix.

56. Obtain the following property of a parabola by projection from a circle :—The line which joins the point of intersection of two tangents to the midpoint of the chord of

contact is parallel to the axis of the parabola, and is bisected by the curve.

57. Generalise by projection :—The locus of the points of contact of tangents drawn from a given point to a system of concentric circles is a circle passing through the given point and the common centre.

58. Two conics S and S′ intersect in the points A, B, C, D ; show that if the pole of AB with regard to S lie on S′, then will the pole of CD with regard to S lie on S′.

59. Tangents are drawn to a conic A parallel to conjugate diameters of a conic B ; prove that they will intersect on a conic C concentric with A, and similar and similarly situated to B.

60. Two sets of points ABC . . ., A′B′C′ . . ., which are homographic but not in perspective, lie on two straight lines ; show that the envelope of AA′, BB′, etc., is a conic touching the two straight lines.

61. Through a point O are drawn chords OAA′, OBB′, OCC′, ODD′, etc., of a conic, and AB′, AC′, AD′ . . ., A′B, A′C, A′D . . . are joined ; prove that the pencils at A and A′ are homographic, and that the pencils at A and O are not homographic unless O is on the conic.

62. A number of conics are inscribed in a given triangle so as to touch one of its sides at a given point ; show that their points of contact with the other two sides form two homographic ranges which are in perspective.

63. Two ranges A, B . . ., A′, B′ . . . are in perspective with respect to a centre S ; show that if J, I′ respectively on each corresponds to the point at infinity on the other, then

$$JA \cdot I'A' = JB \cdot I'B' = \ldots = JS \cdot I'S.$$

If S is equidistant from the two axes, and they are turned about their point of intersection (so as to diminish the angle in which S lies) until they coincide, what is the nature of the system so formed? Determine its centre and double points.

64. Prove, by projecting confocal conics, that the locus of the poles of a given straight line with respect to all conics inscribed in the same quadrilateral is another straight line.

65. Construct the double lines of two homographic pencils having the same vertex.

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